# A Class of Piecewise-Linear Basis Functions and Piecewise-Linear Signal Decomposition and Synthesis

# A Class of Piecewise-Linear Basis Functions and Piecewise-Linear Signal Decomposition and Synthesis

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A set of piecewise-linear basis functions for signal decomposition and synthesis is introduced. These functions span the space $L^2$ [0, 1], the space of functions square integrable on a finite interval [0, 1]. Any signal in $L^2$ [0, 1] can be expanded in terms of these functions with finite term approximations giving a piecewise-linear representation of the signal. The determination of the expansion coefficients is quite simple, and no time domain multiplications or integrations are required. Efficient algorithms for fast computation of the piecewise-linear signal decomposition and synthesis are also presented.

## 1. Introduction

Piecewise-linear (abbreviated to PL) signal decomposition/synthesis technique provides a useful means for a wide variety of engineering applications including signal processing, data compression, simulation, and control.

Paul and Koch have proposed[1] a set of basis functions for PL decomposition and synthesis of continuous functions. The basis set is composed of a sequence of subsets. The $k$th subset contains $2^{k-1}$ functions. Paul's PL series should be truncated so the series contains all the terms included in the same subset. Improper grouping causes an additional truncation error. This means that if the accuracy for the series truncated after the Nth subset is not satisfactory, one must add $2^N$ more terms to improve the accuracy.

To cover this disadvantage, we introduce another class of PL basis functions derived from Haar functions via integration and discuss some properties of the functions. Then we discuss the PL signal decomposition/synthesis using these basis functions and develop efficient computational algorithms.

## 2. Haar Functions and Corresponding PL Basis Functions

Haar functions are a complete orthonormal basis of rectangular waveform and were proposed originally by Haar[2] in 1910. The Haar functions have three possible states: 0, +A, and −A, where A is a constant which is a function of $\sqrt{2}$, and are defined for $t \in [0, 1]$ by:

$$\mathrm{har}\,(0,0,t) = 1, \qquad 0 \leqslant t \leqslant 1,$$

$$\mathrm{har}\,(k,i,t) = \begin{cases} 2^{(k-1)/2}, & (2i-2)/2^k \leqslant t \leqslant (2i-1)/2^k, \\ -2^{(k-1)/2}, & (2i-1)/2^k \leqslant t \leqslant 2i/2^k, \\ 0, & \text{elsewhere}, \end{cases} \tag{1}$$

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where $k = 1, 2, ...,$ and $i = 1, 2, ..., 2^{k-1}$. The Haar functions are labeled by two indices $k$ and $i$. We shall call, hereafter, the indices $k$ and $i$ as the group index and the subindex, respectively, of the Haar and corresponding PL functions. The first eight of the Haar functions are illustrated in Fig. 1a.

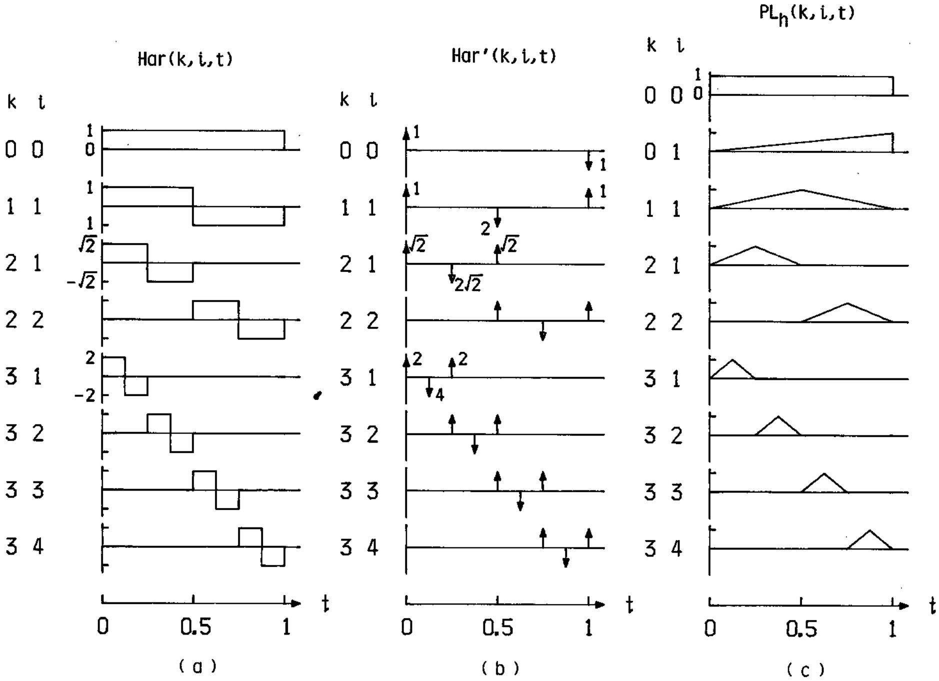


Fig. 1 First eight of Haar functions (a), their derivatives (b), and corresponding $PL_h$ functions (c).

From the Haar functions, a new class of piecewise-linear basis function set is derived. To keep the proposed functions distinct from Paul's PL functions, we shall denote the new functions by $PL_h(k, i, t)$. The subscript h implies that the new functions are derived from Haar functions. Thus the new $PL_h$ basis functions are defined over a unit interval [0, 1] as:

$$\left.\begin{aligned} &PL_h(0, 0, t) = 1, \\ &PL_h(0, 1, t) = \int_0^t har(0, 0, \tau)\, d\tau, \\ &PL_h(k, i, t) = 2^{(k+1)/2} \int_0^t har(k, i, \tau)\, d\tau, \end{aligned}\right\} \quad (2)$$

$$k = 1, 2, ..., \quad i = 1, 2, ..., 2^{k-1}.$$

The multiplicative factor $2^{(k+1)/2}$ in the above equation normalizes the peak of $PL_h(k, i, t)$ to unity. The function $PL_h(k, i, t)$ has a triangular waveform in the subinterval $[(i-1)/2^{k-1}, i/2^{k-1}]$, elsewhere it equals zero, while the Paul's PL functions are composed of a train of triangular pulses. The first nine of the $PL_h$ functions are shown in Fig. 1c.

## 3. Completeness

It is a direct consequence of the completeness of the Haar basis set in the space of square integrable functions that the set of the $PL_h$ functions forms a complete basis in the space of continuous functions. In the following, we show that these functions are also complete in the space of square integrable functions as well as in the space of continuous ones.

Suppose that $\phi(t)$ is a function square integrable in [0, 1] for which

$$\int_0^1 \phi(t)\, PL_h(k, i, t)\, dt = 0, \tag{3}$$

for every $PL_h$ function. Substituting Eq.(2) into Eq.(3), we obtain

$$\begin{aligned} &\int_0^1 \phi(t)\, dt = 0, && k = i = 0, \\ &\int_0^1 \phi(t) \left[\int_0^t \mathrm{har}(k, i, \tau)\, d\tau\right] dt = 0, && \text{elsewhere.} \end{aligned} \tag{4}$$

Integrating Eq.(4) by parts and using Eq.(3), we obtain

$$\int_0^1 \Phi(t)\, \mathrm{har}(k, i, t)\, dt = 0, \tag{5}$$

with

$$\Phi(t) = \int_0^t \phi(\tau)\, d\tau. \tag{6}$$

The function $\Phi(t)$ is continuous over [0, 1] because $\phi(t) \in L^2\,[0, 1]$. Then $\Phi(t)$ can be approximated arbitrarily closely in the sense of uniform convergence by a truncated sum of the Haar-Fourier series $H_N(t)$ when the number of terms is large enough. Thus we can write for an $\epsilon > 0$, however small,

$$|\Phi(t) - H_N(t)| < \epsilon, \tag{7}$$

for all $t \in [0, 1]$. From Eqs.(5) and (7), we obtain

$$\left|\int_0^1 \Phi^2(t) dt\right| = \left|\int_0^1 \Phi(t)\,[\Phi(t) - H_N(t)]\, dt\right| \leqslant \epsilon \int_0^1 |\Phi(t)|\, dt. \tag{8}$$

Since $\epsilon$ can be made arbitrarily small, we have

$$\left|\int_0^1 \Phi^2(t)\, dt\right| = 0. \tag{9}$$

This leads to

$$\Phi(t) \equiv 0, \tag{10}$$

because of the continuity of $\Phi(t)$. Thus we arrive at

$$\phi(t) = \Phi'(t) = 0, \text{ a.e.} \tag{11}$$

This proves that the $PL_h$ functions form a complete basis for the space of functions square integrable over the unit interval [0, 1].

## 4. $PL_h$-Series Signal Decomposition and Synthesis

Any function square integrable on a unit interval [0, 1] can be expanded into an infinite series in terms of the $PL_h$ functions:

$$f(t) = C_{00} + C_{01}\mathrm{PL_h}(0, 1, t) + \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} C_{ki}\mathrm{PL_h}(k, i, t), \tag{12}$$

in which the expansion coefficients are given by

$$C_{00} = f(0), \tag{13a}$$

$$C_{01} = -\int_0^1 f(t)\, \mathrm{har}'(0, 0, t)\, \mathrm{d}t, \tag{13b}$$

$$C_{ki} = -2^{-(k+1)/2} \int_0^1 f(t)\, \mathrm{har}'(k, i, t)\, \mathrm{d}t, \tag{13c}$$

$$k = 1, 2, ..., \text{and } i = 1, 2, ..., 2^{k-1},$$

where the prime denotes the derivative in the usual delta function sense.

Equation (13a) is immediate from the property of the $PL_h$ functions that $\mathrm{PL_h}(k, i, t) = 0$ for $t = 0$ except the case $k = i = 0$ where it becomes 1. To prove Eqs.(13b) and (13c), we simply multiply Eq.(12) by $\mathrm{har}'(l, j, t)$ with the substitution of Eq.(2) and integrate over [0, 1] to yield

$$\begin{aligned} &\int_0^1 f(t)\, \mathrm{har}'(l, j, t)\, \mathrm{d}t \\ &= C_{00} \int_0^1 \mathrm{har}'(l, j, t)\, \mathrm{d}t + C_{01} \int_0^1 \left[\int_0^t \mathrm{har}(0, 1, \tau)\, \mathrm{d}\tau\right] \mathrm{har}'(l, j, t)\, \mathrm{d}t \\ &+ \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} 2^{(k+1)/2} C_{ki} \int_0^1 \left[\int_0^t \mathrm{har}(k, i, \tau)\, \mathrm{d}\tau\right] \mathrm{har}'(l, j, t)\, \mathrm{d}t. \end{aligned} \tag{14}$$

Integrating by parts through the use of the orthogonality of Haar functions, we arrive at Eqs.(13b) and (13c).

Note that $\mathrm{har}'(k, i, t)$ is a triplet of delta functions arising at $t = (i-1)/2^{k-1}$, $\{i-(1/2)\}/2^{k-1}$, and $i/2^{k-1}$ with alternating signs and weights of magnitudes $2^{(k-1)/2}$ for $t = (i-1)/2^{k-1}$ and $i/2^{k-1}$, and $2^{(k+1)/2}$ for $t = \{i-(1/2)\}/2^{k-1}$, except the case $\mathrm{har}'(0, 0, t)$ where it is a pair of delta functions with alternating signs and weight of magnitude one. Therefore Eqs.(13a) through (13c) are reduced to:

$$C_{00} = f(0), \tag{15a}$$

$$C_{01} = f(1) - f(0), \tag{15b}$$

$$C_{ki} = f\left[\{i-(1/2)\}/2^{k-1}\right] - \frac{1}{2}\left[f\{(i-1)/2^{K-1}\} + f(i/2^{k-1})\right],$$

$$k = 1, 2, \ldots, \quad i = 1, 2, \ldots, 2^{k-1}. \tag{15c}$$

Next we discuss some properties of the $PL_h$ series.

**Property 1**: The partial sums of the $PL_h$ series of length not exceeding $2^N + 1$, where N is a positive integer, are piecewise-linear and continuous in [0, 1] with break-points at most at only $2^N + 1$ equidistant binary rational points $j/2^N$, $j = 0, 1, \ldots, 2^N$.

The proof is immediate from the inspection of the waveforms of $PL_h$ functions.

**Property 2**: The partial sums of the first $2^N + 1$ terms of the $PL_h$ series for a continuous function $f(t)$ give a PL approximation to the function which takes on its function value at every $2^N + 1$ equidistant binary rational point $t = j/2^N$, $j = 0, 1, \ldots, 2^N$, and joins these values successively with linear segments.

To prove this, we use the mathematical induction. Let $S_N$ be a $2^N + 1$ length partial sum of the infinite series Eq.(12).

**[step 1]** For the case N = 1, Eq.(12) is reduced to

$$S_1(t) = C_{00} + C_{01}\,PL_h(0, 1, t) + C_{11}\,PL_h(1, 1, t). \tag{16}$$

This leads to

$$\left.\begin{aligned} &S_1(0) = f(0), \\ &S_1(1/2) = f(1/2), \\ &S_1(1) = f(1). \end{aligned}\right\} \tag{17}$$

**[step 2]** Assume that, for an arbitrary N, the partial sum

$$S_N(t) = C_{00} + C_{01}\,PL_h(0, 1, t) + \sum_{k=1}^{N} \sum_{i=1}^{2^{k-1}} C_{ki}\,PL_h(k, i, t), \tag{18}$$

which contains $2^N + 1$ terms, takes on the value of $f(t)$ at every binary rational point $j/2^N$, $j = 0, 1, \ldots, 2^N$. Then $S_{N+1}$, which contains $2^N$ more terms than $S_N$, namely all $PL_h$ basis functions with the group index N + 1, takes on the value of $f(t)$ at every even numbered binary rational point $2j/2^{N+1}$ because all $PL_h$ functions with the group index $k$ = N + 1 are zero at these even numbered binary rational points. For odd numbered binary rational points $(2j+1)/2^{N+1}$, the partial sum $S_{N+1}$ is reduced to

$$S_{N+1}\,[(2j + 1)/2^{N+1}] = S_N\,[(2j + 1)/2^{N+1}] + C_{N+1,j+1}, \tag{19}$$

because

$$PL_h\,[N + 1, i, (2j + 1)/2^{N+1}] = \begin{cases} 1, & j = i - 1, \\ 0, & j \neq i - 1. \end{cases} \tag{20}$$

From the **Property 2**, we can write for the odd numbered binary rational points

$$S_N\,[(2j+1)/2^{N+1}] = [f(2j/2^{N+1}) + f\{(2j+2)/2^{N+1}\}]/2. \tag{21}$$

Substituting Eqs.(15c) and (21) into Eq.(19), we obtain

$$S_{N+1}\,[(2j+1)/2^{N+1}] = f\,[(2j+1)/2^{N+1}], \tag{22}$$

where $j = 0, 1, ..., 2^N - 1$. Thus we arrive at the conclusion that for any N, the partial sum $S_N(t)$ takes on the value of $f(t)$ at every binary rational point $j/2^N$, $j = 0, 1, ..., 2^N$. This, together with the **Property 1**, completes the proof.

**Property 3**: Let S be a partial sum of the infinite series Eq.(12). Let S take on the values of $f(t)$ at two succeeding binary rational points $t_a = (j-1)/2^k$ and $t_b = j/2^k$, where $k$ is an arbitrary positive integer and $j = 1, 2, ..., 2^k$. Adding one more term $C_{k+1,j} \cdot \mathrm{PL_h}(k+1, j, t)$ to the series S, the resultant series exactly takes on the value of $f(t)$ at $(t_a+t_b)/2$, the center of the subinterval $[t_a, t_b]$, as well as at $t_a$ and $t_b$, and joins these values with linear segments without changing the values of the sum outside the subinterval.

The proof is immediate from Eq.(15c) and the property of the $\mathrm{PL_h}$ functions that $\mathrm{PL_h}(k+1, i, t) = 0$ for $t = (i-1)/2^k$ and $i/2^k$, and it equals 1 for $t = [i - (1/2)]/2^k$.

From the properties described above, we obtain the following theorems concerning to the convergence behavior of the $\mathrm{PL_h}$ series.

**Theorem 1**: For continuous functions, the sequence of the finite sums of the $\mathrm{PL_h}$ series of length n will uniformly converges to the functions as $n \to \infty$.

**Theorem 2**: For piecewise-continuous functions with a finite number of jump discontinuities, the $\mathrm{PL_h}$ series, while no longer uniformly convergence, is pointwise convergent everywhere except at the discontinuities.

## 5. Efficient Computational Algorithms

### 5.1. PLh signal decomposition

Sampling of the Haar functions in Fig. 1a at eight equidistant points results in a 8 X 8 matrix as shown in Fig. 2. In general, a $2^N \times 2^N$ matrix is obtained. Each row of the matrix so obtained gives the $2^N$-length discrete representation of the corresponding Harr function and is denoted by $\mathrm{HAR_N}(k, i, j)$, $j = 0, 1, ..., 2^N - 1$.

$\mathrm{HAR_N}(K,i,j) =$

| j = | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | k= | i= |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 |
| | √2 | √2 | -√2 | -√2 | 0 | 0 | 0 | 0 | 2 | 1 |
| | 0 | 0 | 0 | 0 | √2 | √2 | -√2 | -√2 | 2 | 2 |
| | 2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 1 |
| | 0 | 0 | 2 | -2 | 0 | 0 | 0 | 0 | 3 | 2 |
| | 0 | 0 | 0 | 0 | 2 | -2 | 0 | 0 | 3 | 3 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 2 | -2 | 3 | 4 |

Fig. 2 Matrix representation of discrete Haar functions.

Using the discrete representation of Haar functions, we can write, for $k \leqslant N$,

$$\mathrm{har}'(k, i, t) = [\mathrm{HAR_N}(k, i, j) - \mathrm{HAR_N}(k, i, j-1)]\ \delta\ [t - (j/2^N)], \qquad (23)$$

$j = 0, 1, \ldots, 2^N$. Substituting Eq.(23) into Eq.(13c) and rearranging, we obtain

$$C_{ki} = 2^{-(k+1)/2} \sum_{j=0}^{2^N-1} \mathrm{HAR_N}(k, i, j)\ [f\{(j+1)/2^N\} - f(j/2^N)], \qquad (24)$$

for $k = 1, 2, \ldots, N$. Equation (24) shows that the expansion coefficients, except the case $k = 0$, can be obtained as the Haar transform of the differences of the two adjacent samples of $f(t)$. This can be performed efficiently by the use of a fast Haar transform algorithm. A flow diagram based on Eq.(24) is illustrated in Fig. 3, for the case $N = 3$, for example. There are several different algorithms for fast Haar transform. Any one of them can be used for the present purpose, but we have used Andrews'[3]. In Fig. 3 the solid lines indicate that the values at the preceding node is to be carried forward to the addition at the next node with the same sign, and the dotted lines indicate that the values are carried forward with sign inverted. The symbol $\otimes^m$ denotes the multiplication by m and $f_*$ stands for $f(*/8)$. The first step of the diagram computes the differences of the adjacent samples and the succeeding three steps, N-steps in general, accomplish the Haar transform of the differences. All values resulted, other than $C_{00}$ and $C_{01}$, are multiplied by 1/2 to obtain the $PL_h$ coefficients. The multiplication by 1/2 can be efficiently accomplished by "one bit shift" operation instead of the multiplication. It should be noted that at each step in the calculation except the first and second steps, half of the nodes require no further calculations. Thus the total number of additions and subtractions is

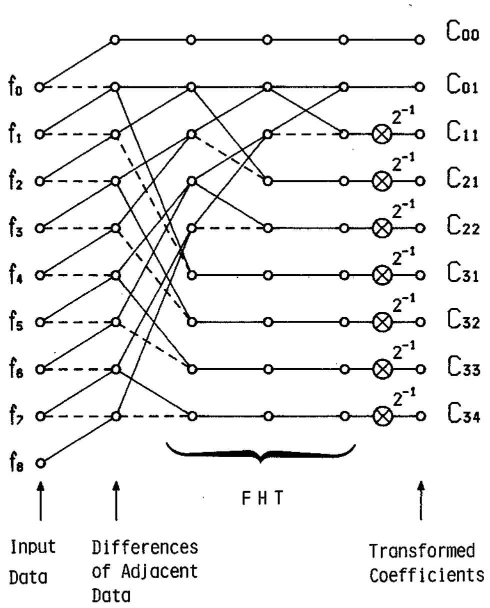


Fig. 3 Signal flow diagram for nine-point $PL_h$ signal decomposition.

$$2^N + 2^N + 2^{N-1} + 2^{N-2} + \cdots + 2^2 + 2^1 = 3 \times 2^N - 2, \tag{25}$$

for $2^N + 1$ samples. This means that, apart from the $2^N - 1$ multiplications in the last step, the computational time for this algorithm is linearly proportional to the number of the samples $2^N$ rather than $N2^N$ as in the case of fast Fourier transform. The algorithm, however, requires additional storage to hold the intermediate stage calculations. To avoid this computational overhead, we can use the "in-place" algorithm proposed by Roeser and Jernigan[4], for instance, but it will give the coefficients arranged in a different order and a sorting routine will be necessary after transformation.

### 5.2. $PL_h$ signal synthesis

Sampling values of the $PL_h$ functions with the group index $k \leqslant N$ at $2^N$ equidistant binary rational points $j/2^N, j = 0, 1, \ldots, 2^N$, are given by

$$PL_h(k, i, j/2^N) = \begin{cases} 1, & k = 0, i = 0, \\ 2^{-N} j, & k = 0, i = 1, \\ 2^{-[N-(k+1)/2]} \sum_{r=0}^{j-1} HAR_N(k, i, r), & \text{elsewhere.} \end{cases} \tag{26}$$

From the **Property 2**, we can write

$$f(j/2^N) = C_{00} + C_{01} PL_h(0, 1, j/2^N) + \sum_{k=1}^{N} \sum_{i=1}^{2^{k-1}} C_{ki} PL_h(k, i, j/2^N). \tag{27}$$

Substituting Eq.(26) into Eq.(27), we obtain

$$f(j/2^N) = C_{00} + C_{01}\, j/2^N + \sum_{r=0}^{j-1} \left[ \sum_{k=1}^{N} \sum_{i=1}^{2^{k-1}} 2^{-[N-(k+1)/2]} C_{ki} HAR_N(k, i, r) \right]. \tag{28}$$

In deriving Eq.(28), we have interchanged the order of the summations. The summations over $k$ and $i$ inside the brackets in Eq.(28) can be computed using fast inverse Haar transform algorithm. The signal flow diagram based on Eq.(28) is illustrated in Fig. 4, for the case $N = 3$ for instance. Again we have used Andrews' algorithm. The solid lines indicate additions and the dotted lines subtractions. The first four steps in the flow diagram compute the inverse Haar transform of the weighted coefficient's array and the succeeding eight steps perform the outermost summation over r in Eq.(28). Note that the last eight steps involve one addition only in each step. Apart from the four, $2^{N-1}$ in general, multiplications in the first step, only additions/subtractions are required. The number of the additions/subtractions required for $2^N + 1$ coefficients' array is again $3 \times 2^N - 2$.

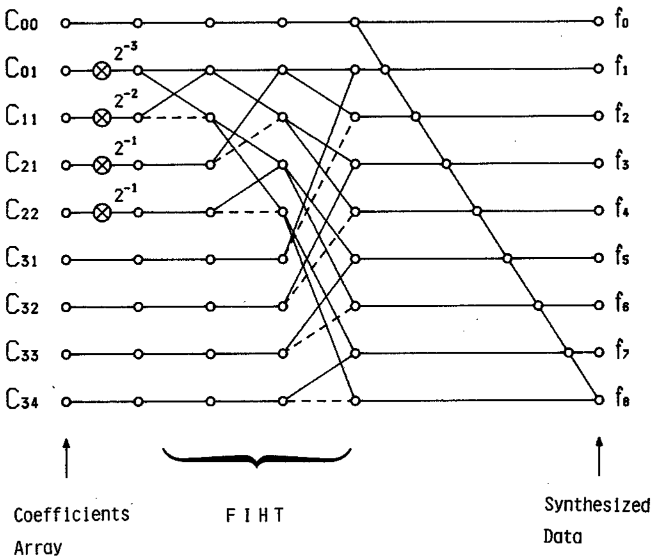


Fig. 4 Signal flow diagram for nine-point $PL_h$ signal synthesis.

## 6. An Example

Consider the quarter period sinusoid shown in Fig. 5 which may represent the output voltage of an ideal thyristor rectifier with resistive load for the ignition angle $\pi/2$. In Fig. 5 the solid line shows the eight term approximation using $PL_h(0, 0, t)$ through $PL_h(3, 3, t)$, and the dotted line shows the true curve. The maximum pointwise approximation error is less than 1 percent. An excellent fit is obtained using a small number of terms.

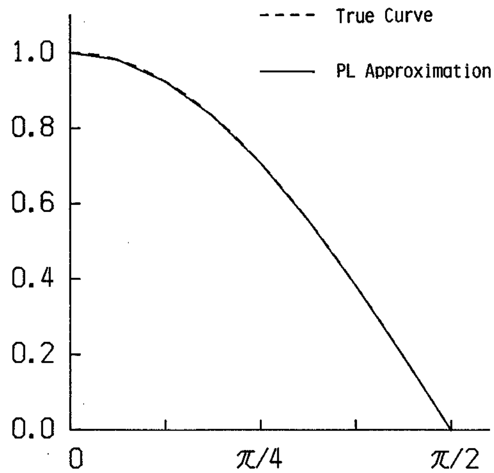


Fig. 5 Finite sum representation of a quarter period sinusoid.

## 7. Conclusion

A set of basis functions has been derived from Haar functions via integration for piecewise-linear signal decomposition and synthesis. The set $\{PL_h(k, i, t)\}$ is a complete basis of $L^2[0, 1]$, the space of functions square integrable over a unit interval $[0, 1]$. Any signal on $L^2[0, 1]$ can be expanded into a series of these basis functions with finite term sums giving a PL approximation to the signal. Efficient algorithms for the $PL_h$ signal decomposition and synthesis have been developed. The algorithms perform the computations without multiplication except the $2^N - 1$ multiplications in the last step of decomposition algorithm and $2^{N-1}$ multiplications in the first step in the synthesis one, both of which are accomplished by simple bit-shift operation. Thus the $PL_h$ signal decomposition/synthesis can be accomplished much faster than those using Fourier series representation. A further advantage of the new basis set is the local dependency of the expansion coefficients. Variations of the signal values in a small subinterval only affect a limited number of coefficients with a large majority of the coefficients unchanged. This property combined with the easy determination of expansion coefficients gives a useful tool for not only signal decomposition/synthesis but also signal analysis, data compression and others.

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