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A Fuzzy Fault Tree Formulated by a Class of Fuzzy Measures

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A fuzzy fault tree for fault analysis of soft systems is formulated by fuzzy measures of fuzzy sets. Fuzzy measures in this paper are based on t-norms and t-conorms. Measures of level-m fuzzy sets are defined recursively, which represent structure functions of the fault tree.

1. Introduction

We have many difficulties in applying fault tree analysis of mechanical systems to soft systems such as education, traffic, crime, etc. The main reason is that the conventional fault tree is based on binary logic. This paper proposes a fault tree formulated by Sugeno's fuzzy measure[1)].

The key property of the fuzzy measure is monotonicity with respect to set inclusion. A broad class of fuzzy measures based on triangular norms was proposed by Dubois and Prade[2)]. Triangular norms are the semi-group operators of the unit interval, which have been studied by Menger[3)], Ling[4)] and, Schweizer and Sklar[5)], amoung others. In this paper an integral proposed by Schwyhla[6)] and Kruse[7)] is adopted to define t-norm based fuzzy measures of fuzzy sets. By including possibility and necessity measures in the definition, Zadeh's possibility and necessity of fuzzy sets are recovered by the fuzzy integral.

Fuzzy measures are regarded as expressing the grade of importance and an application to the fault tree is presented. The grade of top event is represented as the measure of a level-m fuzzy set. The irrelevency and the coherency of the fault tree are discussed. Utilizing duality of t-norm and t-conorm the dual fault tree can be defined.

2. t-Norm Based Fuzzy Measures

In this section, we briefly survey t-norm based fuzzy measures[2)] and related materials in functional equations.

Definition 2.1 A t-norm T[4)] is a two place real valued function from $[0, 1] \times [0, 1]$ to $[0, 1]$, and which satisfies the following conditions

$$T1: \; T(0,0)=0,\; T(a,1)=T(1,a)=a$$

$$T2: \; a \leqslant c,\, b \leqslant d \rightarrow T(a,b) \leqslant T(c,d)$$

$$T3: \; T(a,b)=T(b,a)$$

$$T4: \; T(a, T(b,c)) = T(T(a,b),c)$$

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T is a semi-group of $[0, 1]$ with identity 1. For any continuous t-norm satisfying the Archimedean property $T5$

$$T5: \ T(a,a) < a, \ \forall a \in (0, 1)$$

there exists a continuous and decreasing function f: $[0, 1] \rightarrow [0, +\infty]$ such that $f(1) = 0$ and

$$T(a,b) = f^*(f(a) + f(b)) \tag{2.1}$$

where f^* is the pseudo-inverse of f, defined by

$$f^*(y) = \begin{cases} f^{-1}(y); & y \in [0, f(0)] \\ 0 & ; \ y \in [f(0), +\infty] \end{cases} \tag{2.2}$$

f is called an additive generator of the t-norm T. Schweizer and Sklar[5] provided a multiplicative representation for strict t-norms as

$$T(a,b) = h^{-1}(h(a) \times h(b)) \tag{2.3}$$

where h: $[0, 1] \rightarrow [0, 1]$ is a strictly increasing function with $h(0) = 0, h(1) = 1$. h is called a multiplicative generator of the strict t-norm. The t-norms generated by functions f such that $f(0) < \infty$ and $f(1) = 0$ is called a nilpotent t-norm. Replacing the condition $T1$ by $S1$

$$S1: \ S(1,1) = 1, \ S(0,a) = S(a,0) = a,$$

a mapping S satisfying $S1$ and $T2$-$T4$ is called a t-conorm.

Definition 2.2 A negation is a one place function C: $[0, 1] \rightarrow [0, 1]$ such that

$C1: \ C(0) = 1$

$C2: \ C$ is involutive i.e. $C(C(a)) = a$

The function C: $[0, 1] \rightarrow [0, 1]$ which satisfies the condition $C2$ is axial symmetric with respect to the straight line $y(x) = x$. Hence the condition $C1$ implies that the function C is a continuous and strictly decreasing function. C is called a Trillas' negation and if T is a t-norm, then $S(a, b) = C[T(C(a), C(b))]$ is a t-conorm[8]. For any negation C there exists a function t: $[0, 1] \rightarrow [0, 1]$ such that $t(0) = 0$, $t(1) = 1$, t is continuous and increasing and

$$C(a) = t^{-1}(1 - t(a)) \tag{2.4}$$

Definition 2.3 Let X be a finite set. A fuzzy measure, in the sense of Sugeno[1] is defined by a set function g from 2^X to $[0, 1]$, such that

1) $g(\phi) = 0, \quad g(X) = 1$
2) $\forall A, B \subset X, \quad \text{if } A \subset B, \quad \text{then } g(A) \leqslant g(B).$

t-Conorm based fuzzy measures are defined by replacing the condition 2) of Definition 2.3 with

$$\forall A, B \subset X, \quad \text{if } A \cap B = \phi, \quad \text{then } g(A \cup B) = g(A) \perp g(B) \tag{2.5}$$

where $\perp$ is a t-conorm. It is easy to see that (2.5) implies the condition 2) of Definition 2.3.

Let C be a negation, then a set function

$$\forall A \subset X, \quad g_c(A) = C(g(\bar{A})) \tag{2.6}$$

defines a t-norm based fuzzy measure. Let $\top$ be a t-norm. If $A \cup B = X$, then

$$g_c(A \cap B) = g_c(A) \top g_c(B) \tag{2.7}$$

holds instead of Eq. (2.5). For $\forall A, B \subset X$,

$$g(A \cup B) \perp g(A \cap B) = g(A) \perp g(B) \tag{2.8}$$

$$g_c(A \cup B) \top g_c(A \cap B) = g_c(A) \perp g_c(B) \tag{2.9}$$

3. Fuzzy Measures of Fuzzy Sets

A t-norm based fuzzy measure is defined by a mapping g: $2^X \to [0, 1]$. In this section we define it by a mapping from a family of all fuzzy subsets $\mathscr{F}(X)$ to $[0, 1]$.

Definition 3.1 Let the membership function of a fuzzy set $\tilde{A}$ be a simple function

$$\mu_{\tilde{A}} = \sum_{i=1}^{n} \mu_i \times \chi_{D_i} \tag{3.1}$$

where $\mu_i \in [0, 1]$ and χ_{D_i} denotes the characteristic function of a set D_i $(D_i \cap D_j = \phi)$. A fuzzy measure of fuzzy set $\tilde{A}$ based on a t-conorm $\perp$ is defined as

$$g(\tilde{A}) = \mathop{\perp}_{i=1}^{n} (\mu_i \top g(D_i)) \tag{3.2}$$

where $\bigcup_{i=1}^{n} D_i = X$, $D_i \cap D_j = \phi \ (i \neq j)$ and, $\perp$ and $\top$ fullfil the restricted distribution law (see Weber[9]).

The restricted distribution law of $\perp$ and $\top$ in Eq. (3.2) is written as

$$a \top (g(D_i) \perp g(D_j)) = (a \top g(D_i)) \perp (a \top g(D_j)) \tag{3.3}$$

$$(g(D_i) \perp g(D_j)) \top a = (g(D_i) \top a) \perp (g(D_j) \top a) \tag{3.4}$$

for $^{\forall}a \in [0, 1]$ and $^{\forall}D_i, D_j \subset X$.

In the remaining part of this paper we assume that $\perp$ is the maximum $\vee$ or a nilpotent t-conorm with normed additive generator t, i.e. normed by $t(1) = 1$. The reasones why we assume as mentioned above are as follows.

An additive generator t of Archimedean t-conorm is unique up to a (positive) multiplicative constant.

In other words, if t_1 and $t_2 = kt_1$, $k > 0$ are additive generators of $\perp_1$ and $\perp_2$ respectively, then

$$a \perp_1 b = a \perp_2 b . \tag{3.5}$$

for all $a, b \in [0, 1]$. The only continuous t-conorms which fullfil the restricted distribution law are $\vee$ and nilpotent t-conorms with normed additive generators.

In the case of $\perp = \vee$, Eqs. (3.3) and (3.4) are valid for any t-norm $\top$. In the cases of nilpotent t-conorms, if a fuzzy measure g is based on a nilpotent t-conorm with a normed additive generator t and g satisfies normalizing condition

$$\sum_{i=1}^{n} (t \circ g)(D_i) = t(1) = 1 \tag{3.6}$$

where $t \circ g$ denotes a composite function of t and g, then the property of Eqs. (3.3) and (3.4) is satisfied by choosing a strict t-norm whose multiplicative generator is t. Schwyhla[6] and Kruse[7] proposed a fuzzy integral for special nilpotent t-conorms, which is the same formula as Eq. (3.2). By the condition $g(X) = 1$, for any finite division D_is, $\perp_{i=1}^{n} g(D_i) = 1$. Hence

$$\sum_{i=1}^{n} (t \circ g)(D_i) \geqslant 1 \tag{3.7}$$

Furthermore if $\sum_{i=1}^{n} (t \circ g)(D_i) = M > 1$, then for $a \in (0, 1)$

$$a \top (\perp_{i=1}^{n} g(D_i)) = a \top 1 = a \tag{3.8}$$

and

$$\begin{aligned} \perp_{i=1}^{n} (a \top g(D_i)) &= t^* (\sum_{i=1}^{n} t(a) \times (t \circ g)(D_i)) \\ &= t^* (t(a) \times M) \\ &\neq a \end{aligned} \tag{3.9}$$

Therefore the condition (3.6) is necessary so that $g(\tilde{A})$ in Eq. (3.2) is well defined for nilpotent t-conorm based fuzzy measures. Equation (3.6) implies that $g \circ m$ is a finite additive measure with $(g \circ m)(X) = 1$, i.e. a probability measure.

Proposition 3.1 Let $\mu_{\tilde{A} \cup \tilde{B}} = \mu_{\tilde{A}} \vee \mu_{\tilde{B}}$ and $\mu_{\tilde{A} \cap \tilde{B}} = \mu_{\tilde{A}} \wedge \mu_{\tilde{B}}$, where $\vee$ and $\wedge$ are maximum and minimum respectively. Fuzzy measures of a fuzzy set $\tilde{A} \in \mathscr{F}(X)$ have following properties.

1) $g(\phi) = 0, g(X) = 1$ and $0 \leqslant g(A) \leqslant 1$.
2) If $\tilde{A} \subset \tilde{B}$ then $g(\tilde{A}) \leqslant g(\tilde{B})$.
3) $g(\tilde{A} \cup \tilde{B}) \perp g(\tilde{A} \cap \tilde{B}) = g(\tilde{A}) \perp g(\tilde{B})$, especially if $\tilde{A} \cap \tilde{B} = \phi$, $g(\tilde{A} \cup \tilde{B}) = g(\tilde{A}) \perp g(\tilde{B})$.
4) If $\tilde{A}$ is a crisp set, then $g(\tilde{A}) \perp g(\bar{\tilde{A}}) = 1$.

Proof. Since $\perp$ and $\top$ are non-decreasing two place functions in unit interval, 1) and 2) are trivial.

3) Let the membership functions of $\tilde{A}$ and $\tilde{B}$ be simple functions as

$$\mu_{\tilde{A}} = \sum_{i=1}^{n} \mu_i \times \chi_{D_i} \tag{3.10}$$

and

$$\mu_{\tilde{B}} = \sum_{i=1}^{n} \nu_i \times \chi_{D_i} \tag{3.11}$$

respectively, where μ_i and $\nu_i \in [0, 1]$.

Since $\perp$ is associative,

$$\begin{aligned} & g(\tilde{A} \cup \tilde{B}) \perp g(\tilde{A} \cap \tilde{B}) \\ &= (\mathop{\perp}_{i=1}^{n} ((\mu_i \vee \nu_i) \top g(D_i))) \perp ((\mathop{\perp}_{n=1}^{n} (\mu_i \wedge \nu_i) \top g(D_i)) \\ &= \mathop{\perp}_{i=1}^{n} (((\mu_i \vee \nu_i) \top g(D_i)) \perp ((\mu_i \wedge \nu_i) \top g(D_i))) \\ &= \mathop{\perp}_{i=1}^{n} ((\mu_i \top g(D_i)) \perp (\nu_i \top g(D_i)) \\ &= (\mathop{\perp}_{i=1}^{n} (\mu_i \top g(D_i))) \perp (\mathop{\perp}_{i=1}^{n} (\nu_i \top g(D_i))) \\ &= g(\tilde{A}) \perp g(\tilde{B}) \end{aligned} \tag{3.12}$$

Hence, if $\tilde{A} \cap \tilde{B} = \phi$ then $g(\tilde{A} \cup \tilde{B}) = g(\tilde{A}) \perp g(\tilde{B})$.

4) If $\tilde{A}$ is a crisp set then $\tilde{A} \cap \bar{\tilde{A}} = \phi$ and $\tilde{A} \cup \bar{\tilde{A}} = X$. Thus, $g(\tilde{A}) \perp g(\bar{\tilde{A}}) = g(X) = 1$.

Let g be a fuzzy measure based on $\perp$, and let C be a negation. We define a set-function for $\forall \tilde{A} \in \mathscr{F}$, as

$$g_c(\tilde{A}) \triangleq C(g(\bar{\tilde{A}})) \tag{3.13}$$

where $\mu_{\bar{\tilde{A}}} = C_0(\mu_{\tilde{A}})$ and C_0 is a negation which is not necessarily equal to C. g_c is also a fuzzy measure with following properties.

Proposition 3.2

1) $g_c(\phi) = 0$, $g_c(X) = 1$ and $0 \leqslant g_c(\tilde{A}) \leqslant 1$.
2) If $\tilde{A} \subset \tilde{B}$ then $g_c(\tilde{A}) \leqslant g_c(\tilde{B})$.
3) $g_c(\tilde{A} \cup \tilde{B}) \top ' g_c(\tilde{A} \cap \tilde{B}) = g_c(\tilde{A}) \top ' g_c(\tilde{B})$

where $\top'$ is a t-norm, the C-dual of t-conorm $\perp$. Especially if $\tilde{A} \cup \tilde{B} = X$ then $g_c(\tilde{A} \cap \tilde{B}) = g_c(\tilde{A}) \top' g_c(\tilde{B})$.

4) If $\tilde{A}$ is a crisp set then $g_c(\tilde{A}) \top' g_c(\bar{\tilde{A}}) = 0$

Proof.

1) Let the membership functions be as in Eqs. (3.10) and (3.11). Since

$$g_c(\tilde{A}) = C(\mathop{\perp}_{i=1}^{n} (C_0(\mu_i) \top g(D_i))) \tag{3.14}$$

if $\tilde{A} = \phi$,

$$g_c(\phi) = C(\mathop{\perp}_{i=1}^{n} (1 \top g(D_i))) = C(1) = 0 \tag{3.15}$$

If $\tilde{A} = X$,

$$g_c(X) = C(\mathop{\perp}_{i=1}^{n} (0 \top g(D_i))) = C(0) = 1. \tag{3.16}$$

2) It follows immediately since $\perp$ and $\top$ are non-decreasing and, C_0 and C are decreasing functions.

3) Since g is based on a $\perp$ and $\top'$ is the C-dual t-norm of $\perp$,

$$\begin{aligned} &C(g_c(\tilde{A} \cup \tilde{B}) \top' g_c(\tilde{A} \cap \tilde{B})) \\ &= C(C(g(\overline{\tilde{A} \cup \tilde{B}})) \top' C(g(\overline{\tilde{A} \cap \tilde{B}}))) \\ &= C(C(g(\bar{\tilde{A}} \cap \bar{\tilde{B}})) \top' C(g(\bar{\tilde{A}} \cup \bar{\tilde{B}}))) \\ &= g(\bar{\tilde{A}} \cap \bar{\tilde{B}}) \perp g(\bar{\tilde{A}} \cup \bar{\tilde{B}}) \\ &= g(\bar{\tilde{A}}) \perp g(\bar{\tilde{B}}) \\ &= C(C(g(\bar{\tilde{A}})) \top' C(g(\bar{\tilde{B}}))) \\ &= C(g_c(\tilde{A}) \top' g_c(\tilde{B})) \end{aligned} \tag{3.17}$$

Hence, $g_c(\tilde{A} \cup \tilde{B}) \top' g_c(\tilde{A} \cap \tilde{B}) = g_c(\tilde{A}) \top' g_c(\tilde{B})$. Especially if $\tilde{A} \cup \tilde{B} = X$ then $g_c(\tilde{A} \cap \tilde{B}) = g_c(\tilde{A}) \top' g_c(\tilde{B})$.

4) If $\tilde{A}$ is a crisp set then $\tilde{A} \cup \bar{\tilde{A}} = X$ and $\tilde{A} \cap \bar{\tilde{A}} = \phi$. Hence $g_c(\tilde{A} \cap \bar{\tilde{A}}) = g_c(\tilde{A}) \top' g_c(\bar{\tilde{A}}) = 0$

In what follows it is assumed that the negation C is equal to C_0, i.e. $\mu_{\bar{\tilde{A}}} = C(\mu_{\tilde{A}})$.

Proposition 3.3 Let $\top'$ and $\perp'$ be C-dual of $\perp$ and $\top$ respectively. When $g(\tilde{A})$ is written as

$$g(\tilde{A}) = \mathop{\perp}_{i=1}^{n} (\mu_i \top g(D_i)), \tag{3.18}$$

$g_c(\tilde{A})$ can be written as

$$g_c(\tilde{A}) = \mathop{\top'}_{i=1}^{n} (\mu_i \perp' g_c(\bar{D}_i)) \tag{3.19}$$

Proof.

$$g_c(\tilde{A}) = C(g(\bar{\tilde{A}}))$$
$$= C(\perp_{i=1}^{n} (C(\mu_i) \top g(D_i)))$$
$$= \top_{i=1}^{n}{}' (C(C(\mu_i) \top g(D_i)))$$
$$= \top_{i=1}^{n}{}' (\mu_i \perp' C(g(D_i)))$$
$$= \top_{i=1}^{n}{}' (\mu_i \perp' g_c(\overline{D_i})) \tag{3.20}$$

The case where $\tilde{A}$ is a crisp set and D_i is an element $x_i \in X$ is stated by Dubois and Prade[2] and $g(A)$ and $g_c(A)$ are called a t-conorm based fuzzy measure and a t-norm based fuzzy measure respectively. See also remarks in Weber[10] for infinite universe set X.

Proposition 3.4 Let g be a fuzzy measure based on a nilpotent t-conorm $\perp$, then g is also based on a nilpotent t-norm.

Proof. Let t_0 be the additive generator of $\perp$.

$$g(\tilde{A}) = \perp_{i=1}^{n} (\mu_i \top g(D_i))$$
$$= t_0^{-1} (\sum_{i=1}^{n} t_0(\mu_i) \times t_0(g(D_i))) \tag{3.21}$$

Let $t_1 = 1 - t_0$. Since $t_0(g(D_i)) = 1 - t_0(g(\overline{D_i})) = t_1(g(\overline{D_i}))$,

$$g(\tilde{A}) = t_1^{-1}(1 - \sum_{i=1}^{n} (1 - t_1(\mu_i)) \times t_0(g(D_i)))$$
$$= t_1^{-1} (\sum_{i=1}^{n} t_1(\mu_i) \times t_1(g(\overline{D_i})))$$
$$= \top_{i=1}^{n}{}_1 (\mu_i \perp_1 g(\overline{D_i})) \tag{3.22}$$

where $\top_1$ is a t-norm with normed additive generator t_1.

Proposition 3.5 Let C be any negation and g be a fuzzy measure based on a nilpotent t-conorm $\perp$. g_c is based on the t-norm $\top'$ which is a C-dual of $\perp$. Then g_c is also based on a nilpotent t-conorm.

Proof. Though we can readily see it by Proposition 3.4, let us show an another proof so that the relation between g and g_c is clarified.

Let g be a fuzzy measure based on a nilpotent t-conorm whose additive generator is t_0. Then, for any negation C,

$$g_c(\tilde{A}) = C(g(\bar{\tilde{A}}))$$
$$= (C \circ t_0^{-1})(\sum_{i=1}^{n} (t_0 \circ C)(\mu_i) \times t_0(g(D_i))) \tag{3.23}$$

Let $t_2(a) = 1 - (t_0 \circ C)(a)$, then

$$t_2^{-1}(b) = (C \circ t_0^{-1})(1 - b) \tag{3.24}$$

Hence, we have

$$\begin{aligned} g_c(\tilde{A}) &= t_2^{-1}\left(1 - \sum_{i=1}^{n} (1 - t_2(\mu_i)) \times t_0(g(D_i))\right) \\ &= t_2^{-1}\left(\sum_{i=1}^{n} t_2(\mu_i) \times (t_2 \circ t_2^{-1} \circ t_0)(g(D_i))\right) \end{aligned} \tag{3.25}$$

t_0: $[0, 1] \rightarrow [0, 1]$ is an increasing function and C: $[0, 1] \rightarrow [0, 1]$ is a decreasing function. Therefore t_2 is an increasing function from $[0, 1]$ to $[0, 1]$. Thus $g_c(\tilde{A})$ can be written as

$$\begin{aligned} g_c(\tilde{A}) &= \mathop{\perp_2}_{i=1}^{n} (\mu_i \top_2 (C \circ t_0^{-1} \circ (1 - t_0))(g(D_i))) \\ &= \mathop{\perp_2}_{i=1}^{n} (\mu_i \top_2 C(g(\overline{D_i}))) \\ &= \mathop{\perp_2}_{i=1}^{n} (\mu_i \top_2 g_c(D_i)) \end{aligned} \tag{3.26}$$

where $\perp_2$ and $\top_2$ are a t-conorm with additive generator $t_2 = 1 - (t_0 \circ C)$ and a t-norm with multiplicative generator t_2 respectively. Hence, it can be seen that g_c is based on a t-conorm $\perp_2$ with t_2, while at the same time g_c is based on C-dual t-norm $\top'$ of t-conorm $\perp$ with additive generator t_0.

It is easy to see that Propositions 3.4 and 3.5 are true in the case where $\tilde{A}$ is a crisp set A.

Proposition 3.6 If C is generated by t, and a fuzzy measure g is based on a nilpotent t-conorm $\perp$ with generator t, then g is also based on the C-dual of the t-conorm, say the t-norm $\top$, namely $g(\tilde{A}) = g_c(\tilde{A})$.

Proof. Since $\sum_{i=1}^{n} t(g(D_i)) = 1$ is assumed,

$$\begin{aligned} g(\tilde{A}) &= t^{-1}\left(\sum_{i=1}^{n} t(\mu_i) \times t(g(D_i))\right) \\ &= t^{-1}\left(1 - \sum_{i=1}^{n} (1 - t(\mu_i)) \times t(g(D_i))\right) \\ &= C(g(\overline{\tilde{A}})) \\ &= g_c(\tilde{A}) \end{aligned} \tag{3.27}$$

Example 3.1 Let $a \perp b = a \vee b$ and a $\top$ $b = a \wedge b$ where $\vee$ and $\wedge$ are maximum and minimum respectively. $\vee$ and $\wedge$ are distributive binary operations. By Definition 3.3

$$g(\tilde{A}) = \bigvee_{i=1}^{n} (\mu_i \wedge g(D_i)) \tag{3.28}$$

Since for any negation C, $\wedge$ and $\vee$ are the C-duals of $\vee$ and $\wedge$ respectively,

$$g_c(\tilde{A}) = \bigwedge_{i=1}^{n} (\mu_i \vee C(g(D_i))) \tag{3.29}$$

by proposition 3.3. Let $g = \Pi$, then

$$^\forall\tilde{A}, {}^\forall\tilde{B},\ \Pi(\tilde{A} \cup \tilde{B}) = \Pi(\tilde{A}) \vee \Pi(\tilde{B}) \tag{3.30}$$

by 2) and 3) of Proposition 3.1. Zadeh's possibility measures of fuzzy sets are recovered. Furthermore, by 2) and 3) of Proposition 3.2,

$$^\forall\tilde{A}, {}^\forall\tilde{B},\ \mathrm{N}(\tilde{A} \cap \tilde{B}) = \mathrm{N}(\tilde{A}) \wedge \mathrm{N}(\tilde{B}) \tag{3.31}$$

where $\mathrm{N} = g_c$. Necessity measures of fuzzy sets are also recovered.

Proposition 3.7 Let $\mu^\vee$ and $\mu^\wedge$ be the maximum and the minimum value of membership function of fuzzy set $\tilde{A} \subset X$ respectively. Then

$$\mu^\wedge \leqslant g(\tilde{A}) \leqslant \mu^\vee \text{ and } \mu^\wedge \leqslant g_c(A) \leqslant \mu^\vee \tag{3.32}$$

Proof. Assuming that $\mu_{\tilde{A}} = k$ and k is a constant from $[0, 1]$,

$$g(\tilde{A}) = \mathop{\perp}_{i=1}^{n} (k \top g(D_i)) = k \top (\mathop{\perp}_{i=1}^{n} g(D_i)) = k \tag{3.33}$$

Hence, by Proposition 3.1–2)

$$\mu^\wedge = g(\tilde{A}_\wedge) \leqslant g(\tilde{A}) \leqslant g(\tilde{A}_\vee) = \mu^\vee \tag{3.34}$$

where $\mu_{\tilde{A}_\wedge} = \mu^\wedge$ and $\mu_{\tilde{A}_\vee} = \mu^\vee$ are assumed. By the monotone decreasingness of C,

$$C(\mu^\vee) \leqslant g(\bar{\tilde{A}}) \leqslant C(\mu^\wedge) \tag{3.35}$$

where $\mu_{\bar{\tilde{A}}} = C(\mu_{\tilde{A}})$ is assumed. Thus,

$$C(C(\mu^\wedge)) \leqslant C(g(\bar{\tilde{A}})) \leqslant C(C(\mu^\vee)) \tag{3.36}$$

$$\mu^\wedge \leqslant g_c(A) \leqslant \mu^\vee \tag{3.37}$$

4. A Fuzzy Fault Tree for Soft Systems

Fuzzy measures in the sense of Sugeno[1] can be considered as expressing the grade of importance of attributes of evaluating objects. The measures of fuzzy set $g(\tilde{A})$ and $g_c(\tilde{A})$ aggregate membership values by the grade of importance, and they can be seen as the averaging operators in the sense of Dubois and Prade[11] from the property of Proposition 3.7.

Since $g(\tilde{A})$ and $g_c(\tilde{A})$ are mappings from $\mathscr{F}(X)$ to $[0,1]$, they also can be seen as a membership functions of level-2 fuzzy set, i.e. a fuzzy set whose elements are fuzzy sets. Denoting level-m fuzzy set as ^mA and defining recursively, measures of level-m fuzzy set can be written as

$$g\,({}^m\tilde{A}) = \mathop{\perp}_{k=1}^{l}\,(g_{\cdot}\,({}^{m-1}\tilde{A}_k) \top g_k) \tag{4.1}$$

or

$$g_c\,({}^m\tilde{A}) = \mathop{\top'}_{k=1}^{l}\,(g_{\cdot}\,({}^{m-1}\tilde{A}_k) \perp' \overline{g_{c_k}}) \tag{4.2}$$

where $g_{\cdot}$ denotes g or g_c, and g_k and g_{c_k} stand for fuzzy measures assigned to each element ${}^{m-1}\tilde{A}_k$.

In this section we apply these notions to fault tree analysis. Let X denote the set of all basic events x_i. A denotes a subset of X such as a cut set. In the conventional fault tree analysis, the occurence of top event is judged by the occurence of a cut set A.

Now we consider a fuzzy fault tree in which the state of basic event x_i is represented by the membership value $\mu_i \in [0, 1]$ and the state of top event is represented by the measure of level m fuzzy set $g\,({}^m\tilde{A})$. Let the structure function $\Psi \in [0, 1]$ be a function of vector $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)$. The structure function of fuzzy *AND* gate shown in Fig. 4.1 is written as

$$\Psi\,(\boldsymbol{\mu}) = g\,(\tilde{A}) = \mathop{\perp}_{i=1}^{n}\,(\mu_i \top g_i) \tag{4.3}$$

For fuzzy *OR* gate shown in Fig. 4.2,

$$\Psi\,(\boldsymbol{\mu}) = g_c\,(\tilde{A}) = \mathop{\top'}_{i=1}^{n}\,(\mu_i \perp' \overline{g_{c_i}}) \tag{4.4}$$

Furthermore, for fuzzy *NOT* gate shown in Fig. 4.3,

$$\Psi\,(\boldsymbol{\mu}) = C\,(\mu_i) \tag{4.5}$$

OR, *AND* and *NOT* gates defined above fullfil DeMorgan-like property such as

$$\begin{aligned} &NOT\,(x_1\ OR\ x_2\ OR\ \cdots\ OR\ x_n) \\ &\quad = (NOT\,x_1)\ AND\ (NOT\,x_2)\ AND\ \cdots\ AND\ (NOT\,x_n) \end{aligned} \tag{4.6}$$

The structure function of the left hand side of Eq. (4.6) is written as

$$\Psi\,(\boldsymbol{\mu}) = C\,(\mathop{\perp}_{i=1}^{n}\,(\mu_i \top g_i)\,) \tag{4.7}$$

By the duality of $\perp$ and $\top'$, and $\top$ and $\perp'$

$$\Psi\,(\boldsymbol{\mu}) = \mathop{\top'}_{i=1}^{n}\,(C\,(\mu_i) \perp' \overline{g_{c_i}}\,) \tag{4.8}$$

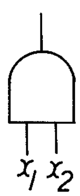

Fig. 4.1 An example of Fuzzy *AND* gate (n=2).

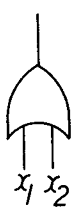

Fig. 4.2 An example of Fuzzy *OR* gate (n=2).

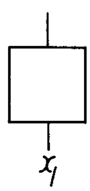

Fig. 4.3 Fuzzy *NOT* gate.

Equation (4.8) implies the right hand side of Eq. (4.6).

By applying three fuzzy gates *OR*, *AND* and *NOT* to a fault tree, a structure function can be obtained as the measure of level m fuzzy set in the form of Eq. (4.1) or Eq. (4.2). The lower level fuzzy sets correspond to the modules of fault tree. Let

$(\cdot_i, \boldsymbol{\mu}) = (\mu_1, \cdots, \mu_{i-1}, \cdot\,, \mu_{i+1}, \cdots, \mu_n)$ and $(a_i, \boldsymbol{\mu}) = (\mu_1, \cdots, \mu_{i-1}, a, \mu_{i+1}, \cdots, \mu_n)$

Definition 4.1 If the structure function $\Psi(\boldsymbol{\mu})$ is invariant with respect to μ_i, that is, for all $a \in [0, 1]$ and $(\cdot_i, \boldsymbol{\mu})$,

$$\Psi(a_i, \boldsymbol{\mu}) = \Psi(1_i, \boldsymbol{\mu}) \tag{4.9}$$

holds, a basic event x_i is said to be irrelevant to the structure Ψ.

In this paper we restrict ourselves to fuzzy measures based on $\vee$ (possibility), $\wedge$ (necessity) and nilpotent triangular conorms and norms so that measures of fuzzy set $g(\tilde{A})$ and $g_c(\tilde{A})$ are well defined. $g(\tilde{A})$ based on $\vee$ is defined for any $\top$ as

$$g(\tilde{A}) = \bigvee_{i=1}^{n} (\mu_i \top g_i) \tag{4.10}$$

And, $g_c(\tilde{A})$ based on $\wedge$ is defined as

$$g_c(A) = \bigwedge_{i=1}^{n} (\mu_i \perp \overline{g_{c_i}}) \tag{4.11}$$

for any $\perp$.

Proposition 4.1 If $^\forall g_i > 0$ and $^\forall g_{c_i} > 0$, then a basic event x_i is irrelevant to the structure Ψ only when all gates in which the event x_i is concerned are based on $\vee$ or $\wedge$.

Proof. By the restricted distributivity of $\perp$ and $\top$, any fuzzy gate which aggregates n items $(n > 2)$ can be represented by a hierarchy of fuzzy gates aggregating only two items. Therefore it is sufficient to show the proof for the case of two level fault tree shown in Fig. 4.4. Let Ψ_1 and Ψ_2 denote the structure functions of the gates G_1 and G_2 in Fig. 4.4 respectively. When the fuzzy gate G_1 is based on $\wedge$ and G_2 is any gate which is not based on $\vee$, there exists $\mu_1 \in (0, 1)$ such that if $\mu_1 > \mu_2$ then $\mu_1 > \Psi_1(\Psi_2, \mu_1)$, while $\mu_1 \leqslant \Psi_1(\Psi_2, \mu_1)$ whenever $\mu_1 \leqslant \mu_2$ by proposition 3.7. Hence, x_2 is relevant to the structure. It is easy to see that x_2 is relevant to the structure in other cases such as

1) G_1 ; based on $\vee$, G_2 ; not based on $\wedge$
2) G_1 ; not based on $\vee$, G_2 ; based on $\wedge$

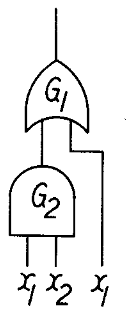

Fig. 4.4 An example of two level fault tree.

3) G_1 ; not based on $\wedge$, G_2 ; based on $\vee$
4) G_1 and G_2 are based on neither $\vee$ nor $\wedge$.

When G_1 is just equal to $\wedge$ and G_2 is equal to $\vee$, i.e. $\forall g_i = 1$ in Eq. (4.10) and $\forall \overline{g_{c_i}} = 0$ in Eq. (4.11), then x_2 is irrelevant.

Definition 4.2 When a structure function $\Psi(\boldsymbol{\mu})$ satisfies the following two conditions, $\Psi(\boldsymbol{\mu})$ is said to be coherent.

1) Each basic event x_i $(i = 1, \cdots, n)$ is relevant to the structure Ψ.
2) $\Psi(\boldsymbol{\mu})$ is non-decreasing with respect to each μ_i $(i = 1, \cdots, n)$.

By the non-decreasingness of t-norms and t-conorms, any structure function of fuzzy fault tree without *NOT* gate is nondecreasing. Hence, if $\boldsymbol{\mu}_1 \geqslant \boldsymbol{\mu}_2$ $(\forall_i, \mu_{1i} \geqslant \mu_{2i})$ then $\Psi(\mu_1) \geqslant \Psi(\mu_2)$. Furthermore $\Psi(\mathit{0}) = 0$, $\Psi(\mathit{1}) = 1$ and

$$\bigwedge_{i=1}^{n} \mu_i \leqslant \Psi(\mu) \leqslant \bigvee_{i=1}^{n} \mu_i \tag{4.12}$$

by Proposition 3.7.

Definition 4.3 A structure function of the dual fault tree Ψ^D is defined as

$$\Psi^D(\boldsymbol{\mu}) = C(\Psi(\boldsymbol{\mu}^C)) \tag{4.13}$$

where $\boldsymbol{\mu}^C = (C(\mu_1), \cdots, C(\mu_n))$.

Replacing *AND* gates by *OR* gates and *OR* gates by *AND* gates we have a dual fault tree whose fuzzy subset of basic events is $\bar{\tilde{A}}$. $\bar{\tilde{A}}$ represents a fuzzy set of successful basic events, whose membership function is $C(\mu_i)$.

Example 4.1 Let $a \perp b = \min\,[1, (a^q + b^q)]^{1/q}$, $a \geqslant 0$. Additive generator of $\perp$ is $t(a) = a^q$ and $t(g(D_i)) = \rho_i$. Then the structure function of *OR* gate is represented by

$$g(\tilde{A}) = (\sum_{i=1}^{n} \mu_i^q \times \rho_i)^{1/q} \tag{4.14}$$

When $C(a) = 1 - a$, *AND* gate is represented by

$$g_c(\tilde{A}) = 1 - (\sum_{i=1}^{n} (1 - \mu_i)^q \times \rho_i)^{1/q} \tag{4.15}$$

Each structure function is reduced to the well known mean values shown in Table 4.1 depending on q.

If $\forall \mu_i \in \{0, 1\}$ and $q \to \infty$ or $q \to 0$ then *OR* gate and *AND* gate are equivalent to *OR* gate or *AND* gate of the conventional fault tree. Similarly by setting as in Eqs. (4.10) and (4.11)

$$g(\tilde{A}) = \bigvee_{i=1}^{n} (\mu_i \top g_i) \tag{4.16}$$

Table 4.1 Mean values assumed by the structure functions in Example 4.1.

	OR gate	*AND* gate
$q \to \infty$	$\bigvee_{i=1}^{n} \mu_i$ maximum	$\bigwedge_{i=1}^{n} \mu_i$ minimum
$q \to 1$	$\sum_{n=1}^{n} \mu_i \times \rho_i$ arithmetic mean	$\sum_{i=1}^{n} \mu_i \times \rho_i$ arithmetic mean
$q \to 0$	$\prod_{i=1}^{n} (\mu_i^{\rho_i})$ When $\forall \rho_i = \frac{1}{n}$, geometric mean	$1 - \prod_{i=1}^{n} (1 - \mu_i)^{\rho_i}$ When $\forall \rho_i = \frac{1}{n}$, dual of geometric mean

and

$$g_c(\tilde{A}) = \bigwedge_{i=1}^{n} (\mu_i \perp \overline{g_{c_i}}), \tag{4.17}$$

where g is a possibility measure and g_c is a necessity measure, we can define *OR* and *AND* gates. It is easy to see that if $\forall g_i = 1$ and $\forall \overline{g_{c_i}} = 0$ then Eqs. (4.1) and (4.2) are conventional structure functions of *OR* gate and *AND* gate respectively.

5. Concluding Remarks

The notion of fuzzy measures of fuzzy sets has been adopted to a fault tree of soft systems. Fuzzy fault tree presented in this paper includes conventional fault tree in the special case. And, fuzzy reliability graph may also be formulated along this line. These attemption will develop wide variety of fault analysis not only for engineering systems but also for social systems.

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