A formulation of Boundary Element Method for Plane Stress Problems

A Formulation of Boundary Element Method for Plane Stress Problems

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The paper presents a formulation of a boundary integral equation for a plane stress problem, based upon Airy's stress function. The integral identity is derived from the weighted residual expression for a biharmonic equation. The necessary boundary integral equations are obtained by selecting a proper singular two-point function as the weighting function in this identity. The sum of the normal stresses is obtained directly from the boundary integral equations. Thus the stress distribution on the boundary is obtained easily. The resulted boundary integral equations are transformed into the simultaneous algebraic equations by the discretization technique. The integrals on each boundary segment (element) are evaluated analytically. The effectiveness of the method is demonstrated by the two numerical examples.

1. Introduction

The integral equation method was already employed as a numerical approach to physical problems in the early part of this century[1]. However the method could not become an useful numerical means in engineerings because of its numerical troublesomeness. The method got renewed in a form of a boundary element method (BEM) in the middle of 1970s[2]. Since then, the method has found its applications in a wide area of engineering[3]. In a stress analysis, a major formulation of BEM is based on a displacement method because of its wide applicability. But the first obtained quantities in this method are displacements and tractions at points on boundaries. The stresses on the boundary have to be computed from the displacements and tractions on the boundary.

The aim of this paper is to present a BEM formulation based on Airy's stress function for plane stress problems. Firstly, the integral identity is derived from the weighted residual expression for a biharmonic equation in a finite region. Secondly, by selecting a proper singular function as a weighting function, necessary boundary integral equations are derived. Finally these equations are extended to an infinite region. One of the advantages of this method is that stresses along the boundary are obtained directly from the boundary integral equations. Numerical applications of the proposed procedure to the extension problems of the infinite region with a hole are given. Variation of a hoop stress along the boundary of typical hole configulations is investigated.

2. Airy's Stress Function

In plane stress problems without body forces, a stress function χ is often introduced to obtain stress components as

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$$\sigma_x = \frac{\partial^2 \chi}{\partial y^2},\ \sigma_y = \frac{\partial^2 \chi}{\partial x^2},\ \tau_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \tag{1}$$

where x, y are Cartesian coordinates. From the compatibility equations, the governing equation of the stress function χ is given as follows;

$$\Delta^2 \chi = 0 \tag{2}$$

where Δ is a Laplacian.

The boundary conditions to χ are given from surface tractions X_n and Y_n as follows;

$$\begin{aligned}
\chi &= \int_A^P \left(\int_B^Q X_n ds\right)dy - \int_A^P \left(\int_B^Q Y_n ds\right)dx + C_1 x + C_2 y + C_3 \\
\frac{\partial \chi}{\partial n} &= \left(\int_A^P X_n ds\right)\cos(y, n) - \left(\int_A^P Y_n ds\right)\cos(x, n) \\
&\quad + C_1 \cos(x, n) + C_2 \cos(y, n)
\end{aligned} \tag{3}$$

Unknown constants C_1, C_2 and C_3 can be put to 0 for simply-connected region. On the other hand, the constants are different each other for multiply-connected region and determined by the conditions of single-valuedness of displacements and rotations[4].

$$\begin{aligned}
&\int_\Gamma \frac{\partial}{\partial n}(\Delta \chi) ds = 0 \\
&\int_\Gamma \left(y\frac{\partial}{\partial n} - x\frac{\partial}{\partial s}\right)\Delta \chi ds = 0 \\
&\int_\Gamma \left(x\frac{\partial}{\partial n} + y\frac{\partial}{\partial s}\right)\Delta \chi ds = 0
\end{aligned} \tag{4}$$

where Γ is the contour of each hole.

3. Boundary Integral Formulation

3.1. Internal problems

The weighted residual representation for the governing equation (2) can be expressed as

$$\int_\Omega (\Delta^2 \chi)\cdot \nu \, d\Omega = 0 \tag{5}$$

where Ω is a region under consideration and ν is a weighting function. Using the divergence theorem, Eq. (5) can be transformed such that derivatives of the stress function are not contained in a region integral

$$\int_\Omega \chi \cdot \Delta^2 \nu d\Omega + \int_\Gamma \left\{ \chi \frac{\partial}{\partial n}(\Delta \nu) - \Delta \nu \cdot \frac{\partial \chi}{\partial n} \right\} d\Gamma$$

$$- \int_\Gamma \left\{ \nu \cdot \frac{\partial}{\partial n}(\Delta \chi) - \Delta \chi \cdot \frac{\partial \nu}{\partial n} \right\} d\Gamma = 0 \qquad (6)$$

where n is an inward normal on a boundary Γ of the region Ω. We can derive necessary boundary integral equations from Eq. (6), selecting a biharmonic function with proper singularity as the weighting function ν in Eq. (6).

Following two-point function is first chosen,

$$\nu(Q,P) = r^2(\ln r - 1), \qquad r = \overline{QP} \qquad (7)$$

where P is an arbitrary fixed point in Ω and Q is any point in the closure $\Omega + \Gamma$ and r is a distance between these points. The function (7) is biharmonic in the region Ω unless the point Q coinsides with the point P (otherwise $r = 0$). Then we consider the region Ω_ϵ removed a circle of a radius ϵ centerd at P from the original region Ω (Fig. 1). In this case, Eq. (6) yields.

$$\int_{\Gamma+C_\epsilon} \left\{ \chi \frac{\partial}{\partial n}(\Delta \nu) - \Delta \nu \frac{\partial \chi}{\partial n} \right\} d\Gamma - \int_{\Gamma+C_\epsilon} \left\{ \nu \frac{\partial}{\partial n}(\Delta \chi) - \Delta \chi \frac{\partial \nu}{\partial n} \right\} d\Gamma = 0 \quad (8)$$

Using the mean value theorem, the integral on the circle C_ϵ becomes

$$\int_{C_\epsilon} (\cdots\cdots) d\Gamma = 8\pi\chi|_Q - 8\pi\epsilon \ln\epsilon \left.\frac{\partial \chi}{\partial n}\right|_Q$$

$$+ 2\pi\epsilon^2(\ln\epsilon^2 - 1) \cdot \Delta\chi|_Q - 2\pi\epsilon^3(\ln\epsilon - 1) \left.\frac{\partial \Delta\chi}{\partial n}\right|_Q \qquad (9)$$

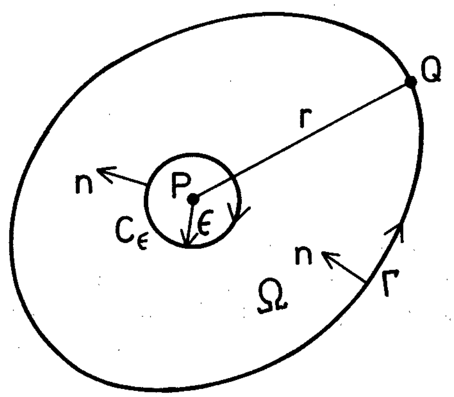

Fig. 1 Field point in a region

where $(\quad)|_Q$ means the value of function at a proper point Q on C_ϵ. The limit $\epsilon \to 0$ in Eq. (9) yields

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} (\cdots\cdots) d\Gamma = 8\pi \chi_P \tag{10}$$

Therefore, the internal value representation is obtained in terms of the boundary quantities as

$$\begin{aligned} 8\pi\chi_p &= 4\int_\Gamma (K_1(P,Q)q_1(Q) + K_2(P,Q)q_2(Q))d\Gamma_Q \\ &+ \int_\Gamma (K_3(P,Q)q_3(Q) + K_4(P,Q)q_4(Q))d\Gamma_Q \end{aligned} \tag{11}$$

where kernel functions $K_i(P,Q)$ are

$$\begin{aligned} &K_1(P,Q) = -\frac{1}{r}\frac{\partial r}{\partial n}, \quad K_2(P,Q) = \ln r \\ &K_3(P,Q) = -\frac{\partial}{\partial n}\left\{r^2(\ln r - 1)\right\}, \quad K_4(P,Q) = r^2(\ln r - 1) \end{aligned} \tag{12}$$

and the boundary functions $q_i(Q)$ are

$$\begin{aligned} &q_1(Q) = \chi_Q\,, && q_2(Q) = (\partial\chi/\partial n)_Q \\ &q_3(Q) = (\Delta\chi)_Q\,, && q_4(Q) = (\partial\Delta\chi/\partial n)_Q \end{aligned} \tag{13}$$

Stresses at any internal points are calculated from Eq. (1) with relation (11).

To obtain the relations between the boundary functions $q_i(Q)$, the point P is taken on the boundary Γ and a half circular region of radius ϵ centered at P is removed from region Ω (Fig. 2). Then Eq. (6) yields

$$\begin{aligned} &4\int_{\Gamma_\epsilon}(K_1q_1 + K_2q_2)d\Gamma + \int_{\Gamma_\epsilon}(K_3q_3 + K_4q_4)d\Gamma \\ &= \frac{4}{\epsilon}\int_{C_\epsilon} q_1 d\Gamma + \epsilon(\ln\epsilon^2 - 1)\int_{C_\epsilon} q_3 d\Gamma \\ &\qquad - 4\ln\epsilon \int_{C_\epsilon} q_2 d\Gamma - \epsilon^2(\ln\epsilon - 1)\int_{C_\epsilon} q_4 d\Gamma \end{aligned} \tag{14}$$

Integrand of the first integral in a left-hand side becomes infinite when $r = 0$, so this integral is an improper integral. On the other hand, the second term include no singularity. Then the integral over Γ_ϵ tends to the integral over Γ as $\epsilon \to 0$. Due to the boundedness of the boundary functions q_i, the right-hand terms take finite values as $\epsilon \to 0$. Finaly, we obtain

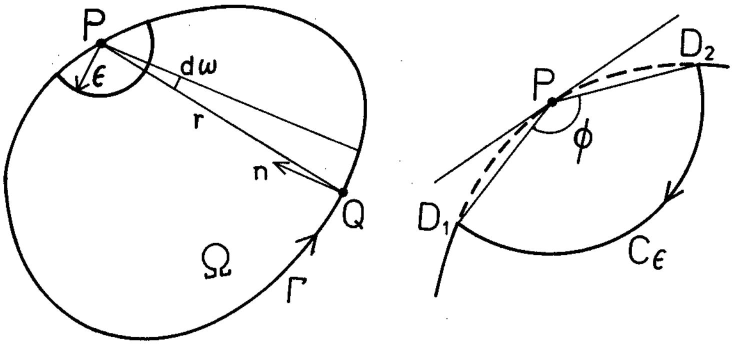

Fig. 2 **Field point on a boundary**

$$4\Phi q_1(P) = 4\int_\Gamma (K_1 q_1 + K_2 q_2) d\Gamma + \int_\Gamma (K_3 q_3 + K_4 q_4) d\Gamma \tag{15}$$

where Φ is an internal angle of contour at P and equal to π if the curve Γ is smooth.

Another relation between boundary quantities can be obtained in a similar manner. As a weighting function, we choose

$$\nu(Q,P) = \ln r\,, \quad r = \overline{QP} \tag{16}$$

Then Eq. (6) yields

$$\Phi\, q_3(P) = \int_\Gamma (K_1 q_3 + K_2 q_4) d\Gamma \tag{17}$$

In Eqs. (15) and (17), boundary values q_1 and q_2 are given from the boundary conditions (3). Then the unknown boundary values q_3 and q_4 are obtained by solving the simultaneous integral equations (15) and (17).

3.2. External problems

In order to treat an external problem, a very good care of the behavior of the stress function at infinity must be taken. From the uniqueness conditions for a biharmonic function defined in an infinite region, function χ must behave as

$$\chi(P) = O(R)\,, \qquad R \to \infty \tag{18}$$

An asymptotic expansion of Eq. (11) in terms of R yields

$$\begin{aligned} 8\pi\chi_p &= R^2(\ln R - 1)\int_\Gamma q_4 d\Gamma + x(\ln R^2 - 1)\int_\Gamma \left(q_3 \frac{\partial \xi}{\partial n} - q_4 \xi\right) d\Gamma \\ &+ y\,(\ln R^2 - 1)\int_\Gamma \left(q_3 \frac{\partial \eta}{\partial n} - q_4 \eta\right) d\Gamma \\ &+ \ln R \int_\Gamma \left(4q_2 - 2q_3 r_Q \frac{\partial r_Q}{\partial n} + q_4 r_Q{}^2\right) d\Gamma + \mathrm{O}(1) \end{aligned} \tag{19}$$

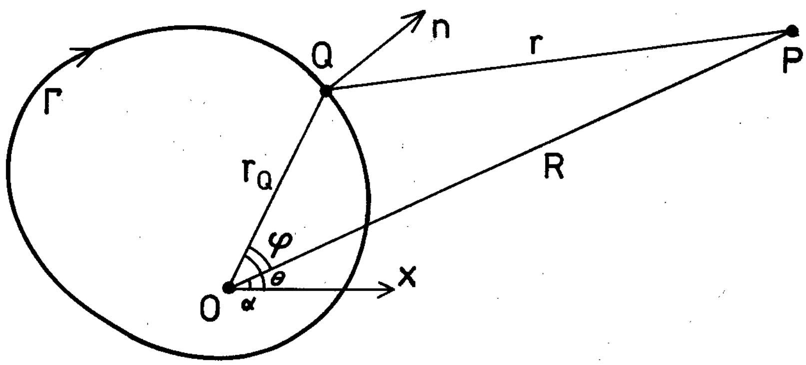

Fig. 3 Notations in an infinite region

where (x, y) and (ξ, η) are Cartesian coordinates to points P and Q, respectively (Fig. 3). To satisfy the condition (18), first three terms must vanish in the right-hand side of Eq. (19). So we obtain

$$\int_\Gamma q_4(Q)d\Gamma = 0$$

$$\int_\Gamma (q_3(Q)\frac{\partial \xi}{\partial n} - q_4(Q)\xi)d\Gamma = 0$$

$$\int_\Gamma (q_3(Q)\frac{\partial \eta}{\partial n} - q_4(Q)\eta)d\Gamma = 0 \tag{20}$$

Since in this situation the terms of $O(R)$ are simultaneously removed, the term $\alpha x + \beta y + \gamma$ have to be added to right-hand side of Eq. (11).

$$\begin{aligned} 8\pi X_P = & 4\int_\Gamma (K_1(P,Q)q_1(Q) + K_2(P,Q)q_2(Q))d\Gamma \\ & + \int_\Gamma (K_3(P,Q)q_3(Q) + K_4(P,Q)q_4(Q))d\Gamma \\ & + \alpha x + \beta y + \gamma \end{aligned} \tag{21}$$

The unknown constants α, β and γ are now to be determined from the conditions (20) which are equivalent to the uniqueness conditions of displacements and rotations around the edge of the holes. For example, Eq. $(20)_2$ is rewritten as follows;

$$\int_\Gamma \left\{(\Delta X)\frac{\partial \xi}{\partial n} - \xi \frac{\partial}{\partial n}(\Delta X)\right\} d\Gamma = \int_\Gamma (\Delta X)\frac{\partial \eta}{\partial s} d\Gamma - \int_\Gamma \xi \frac{\partial}{\partial n}(\Delta X) d\Gamma$$

$$= (\Delta X)\eta \,|_A^A - \int_\Gamma (\xi \frac{\partial}{\partial n} + \eta \frac{\partial}{\partial s})\, \Delta X d\Gamma \tag{22}$$

The first term in the right-hand side of above equation vanishes due to the single-valuedness of stresses. Thus Eq. $(20)_2$ is equivalent to Eq. $(4)_2$.

4. Numerical Treatment

Since it is in general very difficult to solve analytically the simultaneous boundary integral equations (15) and (17), these equations must be discretized and numerically solved. First the boundary is devided into N small segments (elements) as shown in Fig. 4. An integral over an element is evaluated by means of a numerical quadrature, using values of function at given nodal points P_j. Assuming the boundary functions to be constant over each element and taking in turn a point P in Eqs. (15) and (17) to a nodal point P_i, $2N$ algebraic equations are obtained as

$$
\begin{aligned}
4\Phi q_1(P_i) &= \sum_{j=1}^{N}\left\{4F_{ij}q_1(P_j)+4G_{ij}q_2(P_j)+H_{ij}q_3(P_j)+R_{ij}q_4(P_j)\right\}\\
&\quad +\alpha x_{P_i}+\beta y_{P_i}+\gamma\\
\Phi q_3(P_i) &= \sum_{j=1}^{N}\left\{F_{ij}q_3(P_j)+G_{ij}q_4(P_j)\right\}
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
F_{ij} &= \int_{\Gamma_j} K_1(P_i,Q)d\Gamma_Q & G_{ij} &= \int_{\Gamma_j} K_2(P_i,Q)d\Gamma_Q\\
H_{ij} &= \int_{\Gamma_j} K_3(P_i,Q)d\Gamma_Q & R_{ij} &= \int_{\Gamma_j} K_4(P_i,Q)d\Gamma_Q
\end{aligned}
\tag{24}
$$

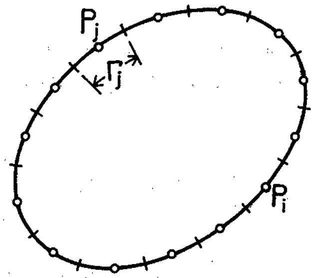

Fig. 4 Subdivisions of a boundary and nodal points

Constants α, β and γ in the above equations are set equal to zero in an internal problem, and determined from additional discretized versions of Eqs. (20) in an external problem. It is difficult to calculate the integrals (24) for a general path. However if the path is a linear segment, these terms can be integrated exactly. Hence the small curved element is replaced with two linear lines which connect the central nodal point and the terminal points of the element as shown in Fig. 5b. The values of each integral are calculated by means of the following analytical results.

[i] Neither point A nor point B coincides with nodal point P_i:

$$
I_1 = \int_A^B K_1(P_i,Q)d\Gamma_Q = \psi
$$

$$
I_2 = \int_A^B K_2(P_i,Q)d\Gamma_Q = a\ln\left(\frac{a}{b}\right)\cos\theta + h(\ln b - 1) + a\psi\sin\theta
$$

$$\int_A^B K_3(P_i, Q)d\Gamma_Q = a\sin\theta\ [2a\ln(\frac{a}{b})\cos\theta + h(\ln b^2 - 3) + 2a\psi\sin\theta]$$

$$\int_A^B K_4(P_i, Q)d\Gamma_Q = \frac{2}{3}\psi a^3\sin^3\theta$$
$$+\frac{h}{3}[(b^2+2a^2\sin^2\theta)(\ln b - \frac{4}{3}) - a^2\sin^2\theta]$$
$$+\frac{a\cos\theta}{3}[(b^2+2a^2\sin^2\theta)\ln\frac{a}{b} + (a^2-b^2)(\ln a - \frac{4}{3})] \quad (25)$$

[ii] Point A or B coincides with nodal point P_i:

$$F_{ii} = \int_{\Gamma_i} K_1(P_i, Q)d\Gamma_Q = \pi - \psi_0$$
$$\int_A^B K_2(P_i, Q)d\Gamma_Q = h(\ln h - 1)$$
$$\int_A^B K_3(P_i, Q)d\Gamma_Q = 0$$
$$\int_A^B K_4(P_i, Q)\,d\Gamma_Q = \frac{1}{3}h^3(\ln h - \frac{4}{3}) \quad (26)$$

where the symbols in the above equations are defined in Fig. 5.

Stresses at an internal point are calculated by using the following equations

$$\sigma_x = \frac{1}{2\pi}\sum_{j=1}^{N}[q_1(P_j)\int_{\Gamma_j}\frac{\partial^2 K_1}{\partial y^2}d\Gamma + q_2(P_j)\int_{\Gamma_j}\frac{\partial^2 K_2}{\partial y^2}d\Gamma]$$
$$+\frac{1}{8\pi}\sum_{j=1}^{N}[q_3(P_j)\int_{\Gamma_j}\frac{\partial^2 K_3}{\partial y^2}d\Gamma + q_4(P_j)\int_{\Gamma_j}\frac{\partial^2 K_4}{\partial y^2}d\Gamma]$$

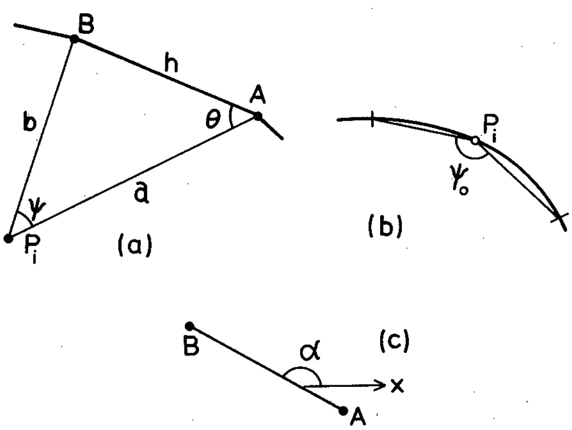

Fig. 5 Notations for a linear element

$$\sigma_y = \frac{1}{2\pi}\sum_{j=1}^{N} [q_1(P_j)\int_{\Gamma_j} \frac{\partial^2 K_1}{\partial x^2} d\Gamma + q_2(P_j)\int_{\Gamma_j} \frac{\partial^2 K_2}{\partial x^2} d\Gamma]$$

$$+ \frac{1}{8\pi}\sum_{j=1}^{N} [q_3(P_j)\int_{\Gamma_j} \frac{\partial^2 K_3}{\partial x^2} d\Gamma + q_4(P_j)\int_{\Gamma_j} \frac{\partial^2 K_4}{\partial x^2} d\Gamma]$$

$$\tau_{xy} = -\frac{1}{2\pi}\sum_{j=1}^{N} [q_1(P_j)\int_{\Gamma_j} \frac{\partial^2 K_1}{\partial x\partial y} d\Gamma + q_2(P_j)\int_{\Gamma_j} \frac{\partial^2 K_2}{\partial x\partial y} d\Gamma]$$

$$-\frac{1}{8\pi}\sum_{j=1}^{N} [q_3(P_j)\int_{\Gamma_j} \frac{\partial^3 K_3}{\partial x\partial y} d\Gamma + q_4(P_j)\int_{\Gamma_j} \frac{\partial^2 K_4}{\partial x\partial y} d\Gamma] \qquad (27)$$

Integrals involved in the above equations are also analytically evaluated for the linear path. The results are as follows;

$$\int_A^B \frac{\partial^2 K_1}{\partial x^2} d\Gamma = -\int_A^B \frac{\partial^2 K_1}{\partial y^2} d\Gamma = (\frac{1}{b^2} - \frac{1}{a^2})\sin 2\alpha$$

$$-2a\sin\theta\, [\frac{h}{b^4}\cos 2\alpha - (\frac{1}{b^4} - \frac{1}{a^4})a\cos(2\alpha + \theta)]$$

$$\int_A^B \frac{\partial^2 K_2}{\partial x^2} d\Gamma = -\int_A^B \frac{\partial^2 K_2}{\partial y^2} d\Gamma = I_7$$

$$\int_A^B \frac{\partial^2 K_3}{\partial x^2} d\Gamma = 2(I_1 - I_5\sin\alpha - I_6\cos\alpha + I_7 a\sin\theta)$$

$$\int_A^B \frac{\partial^2 K_4}{\partial x^4} d\Gamma = 2I_2 + h\cos 2\alpha - 4a\sin\theta\,(I_5\sin\alpha + I_6\cos\alpha)$$

$$\int_A^B \frac{\partial^2 K_3}{\partial y^2} d\Gamma = 2(I_1 + I_5\sin\alpha + I_6\cos\alpha - I_7 a\sin\theta)$$

$$\int_A^B \frac{\partial^2 K_4}{\partial y^2} d\Gamma = 2I_2 - h\cos 2\alpha + 4a\sin\theta(I_5\sin\alpha + I_6\cos\alpha)$$

$$\int_A^B \frac{\partial^2 K_1}{\partial x\partial y} d\Gamma = (\frac{1}{a^2} - \frac{1}{b^2})\cos 2\alpha$$

$$-2a\sin\theta\, [\frac{h}{b^4}\sin 2\alpha - (\frac{1}{b^4} - \frac{1}{a^4})a\sin(2\alpha + \theta)]$$

$$\int_A^B \frac{\partial^2 K_2}{\partial x\partial y} d\Gamma = I_8$$

$$\int_A^B \frac{\partial^2 K_3}{\partial x\partial y} d\Gamma = 2(I_5\cos\alpha - I_6\sin\alpha + I_8 a\sin\theta)$$

$$\int_A^B \frac{\partial^2 K_4}{\partial x \partial y} d\Gamma = h\sin 2\alpha + 4a\sin\theta(I_5 \cos\alpha - I_6 \sin\alpha) \tag{28}$$

where I_1 and I_2 are defined in Eq. (25), and

$$I_5 = \ln(a/b)\cos\alpha - \psi\sin\alpha$$

$$I_6 = \ln(a/b)\sin\alpha + \psi\cos\alpha$$

$$I_7 = (h/b^2)\cos 2\alpha - \left(\frac{1}{b^2} - \frac{1}{a^2}\right) a\cos(2\alpha + \theta)$$

$$I_8 = (h/b^2)\sin 2\alpha - \left(\frac{1}{b^2} - \frac{1}{a^2}\right) a\sin(2\alpha + \theta) \tag{29}$$

where α is the angle between the linear path AB and the positive x-axis. (Fig. 5c).

5. Numerical Examples

5.1. Hoop stress around a circular hole

The stresses around a circular hole are calculated in the infinite region in a state of simple tensile stress S as shown in Fig. 6. This example is one of the most standard problems in two-dimensional elasticity[5] and selected to verify the validity of the proposed method. Numbers of elements are taken as 12, 18, 24 and 30. The numerical results of the tangential stress σ_θ are plotted along the edge in Fig. 7 and compared with the analytical one shown by a solid line. It is seen that 12-element-approximation even gives a good agreement with the exact solution.

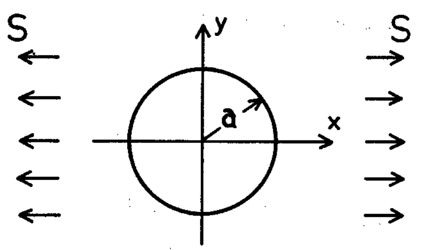

Fig. 6 Circular hole in an infinite plate

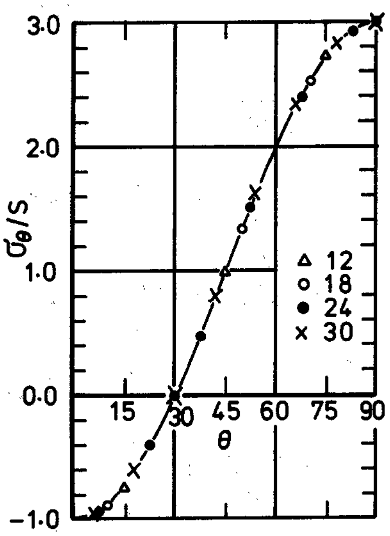

Fig. 7 Circumferential stress along the circular edge

5.2. Hoop stress along a bulged square hole

When a plate with a hole is extended, a high stress concentration occurs around the hole. Since a stress concentration is a bug to the strength of the plate, the stress distribution should be as flattened as possible. Smoothing of the stress distribution is considered for the bulged square hole with rounded corners in the infinite region under the simple tensile stress S as shown in Fig. 8. The corner radius r of the bulged square hole is assumed to be $a/3$, where a is a half length of the side of the square hole. Figure 9 shows the variations of tangential stresses with respect to the height h of top and bottom bulges. It is seen that a high stress concentration occurs at the corner part, when the top and bottom sides do not bulged, that is $h = 0$. As the height of the bulge increases, the stresses at the corner decreases gradually and the one along the bulged part AB increases. The almost flattened distribution of the stress is shown in Fig. 10 with the stress distribution for the non-bulged case. It is observed that the bulge, only 11.32% of a half length of the side, reduces the stress concentration considerably.

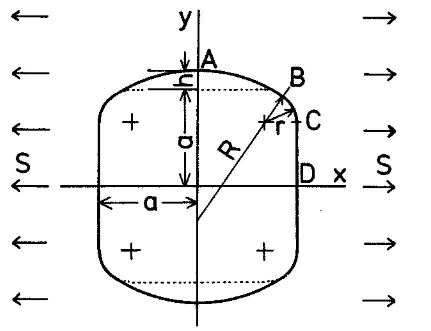

Fig. 8 Bulged square hole in an infinite plate

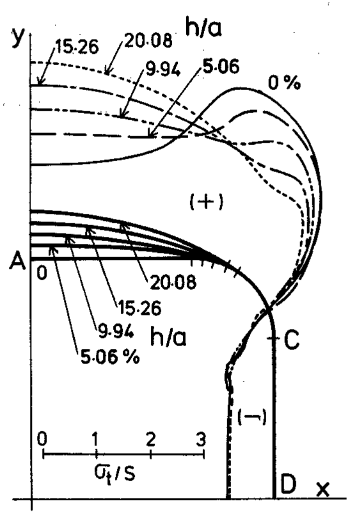

Fig. 9 Variations of tangential stress with respect to the bulge height

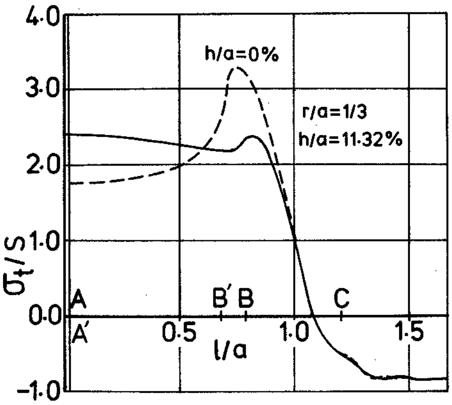

Fig. 10 Flattened stress distribution

6. Concluding Remarks

The present paper has proposed a formulation of BEM based on the stress method. The obtained boundary integral equations are numerically solved after discretizations. The integrals on the boundary elements are evaluated analytically by assuming a pair of linear paths. The expressions for the integrals over the linear path are given in closed forms. The tangential stress on the boundary can be obtained easily, because the unknown quantities q_3 in the boundary relations gives the sum of the normal stresses. It is verified through the examples that the BEM proposed here is useful and effective for solving the plane stress problems.

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