A Method of Optimal Control in Nonlinear Systems

A Method of Optimal Control in Nonlinear Systems

Hiroaki KAWABATA*, Hidekuni HASHIZUME**, Yoshio INAGAKI*,
Yoshiaki SHIRAO*, Toshikuni NAGAHARA* and Masao KIDO*

(Received June 16, 1986)

The derivation of optimal nonlinear feedback control laws for nonlinear systems with non-quadratic performance critearia is presented by using inverse optimum control problem. The nonlinear systems under the consideration are restricted to the systems which are nonlinear with respect to the state variables. The validity of this method is shown by numerical examples.

1. Introduction

It has been long time since the optimal state feedback control law of a linear system that is represented by the L.Q.G. problem has constructed. And it has been reported that these theories are effective in the domain that is approximately linearized in an actual nonlinear system in various fields[1−4)].

However, the nonlinear regulator is essentially demanded in the nonlinear system which shows the remarkable nonlinearity so that various methods of the optimal regulator for the nonlinear systems have been considered. For example, the method approximating the nonlinear system by the connection of several linear systems, and the method using Liapunov function are proposed.

In this paper we propose a new method that asymptotically stabilizes the nonlinear system and minimizes a certain cost function by using a Liapunov function. It will be considered that this method is one improving on the method proposed by Jacobson.

2. Problem Statement

Let us consider the optimal control problem such that the system is nonlinear with respect to the state variables only;

$$\dot{x} = f(x) + Bu \tag{1}$$

where x and u are the $n \times 1$ state vector and the $r \times 1$ control vector, $f(x)$is nonlinear vector valued function of n dimensions which is differential with respect to state vector x, and B is a matrix of appropriate dimensions. Equation (1) is a familiar nonlinear differential equation, for instance, the equation of the transient motion of the generator in power system.

Now we define the cost performance function as

$$J = \int_0^\infty [q(x) + \frac{2k+1}{2(p+k+1)} \sum_{i=1}^{r} u_i^{\frac{2(p+k+1)}{2k+1}}]\, dt \tag{2}$$

* Department of Electrical Engineering, College of Engineering.
** Graduate Student, Department of Electrical Engineering, College of Engineering.

where $q(x)$ is a positive function of the state x, u_i is the i-th component of control vector $\boldsymbol{u}$ and p, k are integer respectively.

It is the purpose of this paper to find the optimal control that minimizes the cost function Eq. (2) in the nonlinear system.

3. Stabilization and Optimal Control

Supposing the positive function $V(x)$ of the state vector x, we consider the following control using the gradient of $V(x)$.

$$\boldsymbol{u} = -\sum_{i=1}^{r} [\{V_x(x)B1_i^r\}^{\frac{2k+1}{2p+1}} \cdot 1_i^r] \tag{3}$$

where $V_x(x)$ denotes the gradient of the Liapunov function $V(x)$, p and k are integer respectively, and 1_i^r is given by

$$1_i^r = [\overbrace{0, 0, \cdots, 0, 1}^{i}, 0, \cdots, 0]^T \quad (\text{length } r) \tag{4}$$

Each components of the vector $\boldsymbol{u}$ are defined by

$$u_i = -[V_x(x)b_i]^{\frac{2k+1}{2p+1}} \quad i = 1, 2, 3, \cdots, r \tag{5}$$

where b_i is the i-th column vector of the matrix B.

(Theorem)

Suppose that there exists a radially unbounded, positive function $V(x)$, such that $V_x(x)f(x)$ is negative semi-difinite. Then the control of Eq. (5) globally asymptotically stabilizes Eq. (1) and minimizes the cost function Eq. (2) in the class of control functions which causes $x(t) \to 0$ as $t \to \infty$.

(Proof)

First, we show that the positive function $V(x)$ satisfies a Liapunov function of the system Eq. (1).

$$\begin{aligned} \dot{V}(x) &= V_x(x) \cdot \dot{x} = V_x(x)\,[f(x) - B\sum_{i=1}^{r} \{(V_x(x)B1_i^r)^{\frac{2k+1}{2p+1}} \cdot 1_i^r\}] \\ &= V_x(x)f(x) - \sum_{i=1}^{r} [V_x(x)b_i]^{\frac{2(p+k+1)}{2p+1}} \end{aligned} \tag{6}$$

Since the $V_x(x)f(x)$ is negative semi-definite and $2(p+k+1)$ is even, $\dot{V}(x) < 0$ and $V(x)$ becomes a Liapunov function of Eq. (1). Control Eq. (5) therefore globally asymptotically stabilizes Eq. (1).

Next, the algebraic Hamilton-Jacobi-Bellman equation for this case becomes as follows

$$\min_{u_i} \left[q(x) + \frac{2k+1}{2(p+k+1)} \sum_{i=1}^{r} u_i^{\frac{2(p+k+1)}{2k+1}} + V_x(x) \left\{ f(x) + \sum_{i=1}^{r} b_i u_i \right\} \right] = 0 \tag{7}$$

By differentiating about u_i, we may write down

$$u_i^{\frac{2p+1}{2k+1}} + V_x(x) b_i = 0 \tag{8}$$

thus the control which minimizes the cost function is given by

$$u_i = -\left[V_x(x) b_i\right]^{\frac{2k+1}{2p+1}} \tag{9}$$

Substituting Eq. (9) into Eq. (7) yields

$$q(x) = -V_x(x) f(x) + \frac{2p+1}{2(p+k+1)} \sum_{i=1}^{r} \left[V_x(x) b_i\right]^{\frac{2(p+k+1)}{2p+1}} \tag{10}$$

Along a solution of Eq. (1) we have

$$V(x_0) - V(x_\infty) + \int_0^\infty \dot{V}(x) dt = 0 \tag{11}$$

so that

$$V(x_0) - V(x_\infty) + \int_0^\infty V_x(x) \left[f(x) + \sum_{i=1}^{r} b_i u_i\right] dt = 0 \tag{12}$$

Adding Eq. (12) to Eq. (2) and substituting (10), there results

$$J = V(x_0) - V(x_\infty) + \int_0^\infty \Big[V_x(x) \left\{ f(x) + \sum_{i=1}^{r} b_i u_i \right\} - V_x(x) f(x) + \frac{2p+1}{2(p+k+1)} \sum_{i=1}^{r} \left[V_x(x) b_i\right]^{\frac{2(p+k+1)}{2p+1}} + \frac{2k+1}{2(p+k+1)} \sum_{i=1}^{r} u_i^{\frac{2(p+k+1)}{2k+1}} \Big] dt \tag{13}$$

Since we chose u_i from the class of functions which causes $x(t) \to 0$ as $t \to \infty$, $V(x(\infty)) = 0$ and the controls Eq. (9) minimize the integral in Eq. (13) to give its minimum value of zero. Hence $V(x_0)$ becomes the minimum value of J.

$$J_{\min} = V(x_0) \tag{14}$$

4. Numerical Examples and Discussion

This principle of the optimal control is valid for essentially nonlinear systems. To show this, we present the simplest interesting system. Our analytical examples are rather elementary, whereas these examples do not carry over into more compelx systems, the behavior is in many ways similar to that observed in the more complex cases.

(Example 1)

Consider the second order nonlinear system

$$\dot{x} = f(x) + Bu \tag{15}$$

where $x = (x_1, x_2)^T, f(x) = (-\frac{1}{2} \sin 2(x_1 - x_2) - x_1, \frac{1}{2} \sin 2(x_1 - x_2))^T$

$$B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, u = (u_1, u_2)^T$$

In terms of this matrix we may then express the Liapunov function for this system by the equation

$$V(x) = \frac{1}{2} \sin^2 (x_1 - x_2) + \frac{1}{2} x_1^2 \tag{16}$$

when $p = 1, k = 1$, the control which gives an asymptotically stable solution for Eq. (15) is written as

$$\begin{aligned} u &= -\sum_{i=1}^{2} V_x(x) [B1_i^2] 1_i^2 \\ &= \begin{bmatrix} -3 \sin (x_1 - x_2) \cos (x_1 - x_2) - 2x_1 \\ 2 \sin (x_1 - x_2) \cos (x_1 - x_2) + x_1 \end{bmatrix} \end{aligned} \tag{17}$$

Calculating dV/dt gives

$$\begin{aligned} \dot{V}(x) = &- [\frac{1}{2} \sin 2(x_1 - x_2) + x_1]^2 - \frac{1}{4} \sin^2 2(x_1 - x_2) \\ &- [\frac{3}{2} \sin 2(x_1 - x_2) + 2x_1]^2 - [\sin 2(x_1 - x_2) + x_1]^2 \leqslant 0 \end{aligned} \tag{18}$$

It is evident that the origin is a stable equilibrium point by using the nonlinear feedback control Eq. (17). And this control Eq. (17) also minimizes the following cost function.

$$J = \int_0^\infty [\frac{15}{4} \sin^2 2(x_1 - x_2) + 9x_1 \sin 2(x_1 - x_2) + 6x_1^2] \, dt \tag{19}$$

Figure 1 illustrates the phase diagram of x_1 and x_2 showing how they start from initial state (1.0, –1.0) for two cases of the optimal nonlinear feedback control and no control. It will be seen that in the case of no control both x_1 and x_2 diverge, but in the

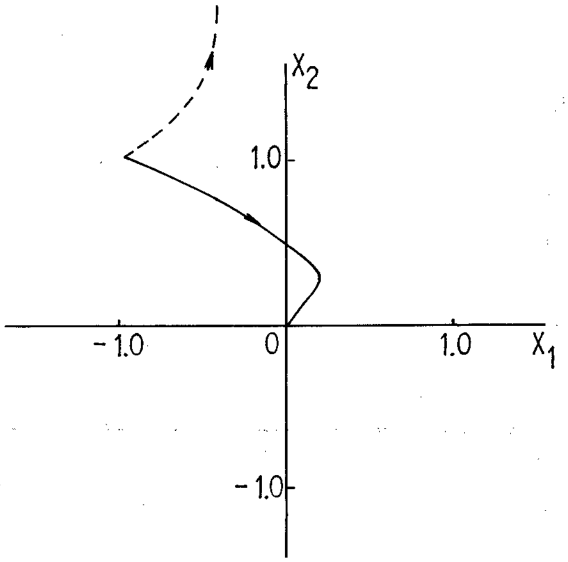

Fig. 1 Phase diagram of x_1 and x_2 (Dotted line is for no control, solid line is for optimal control.).

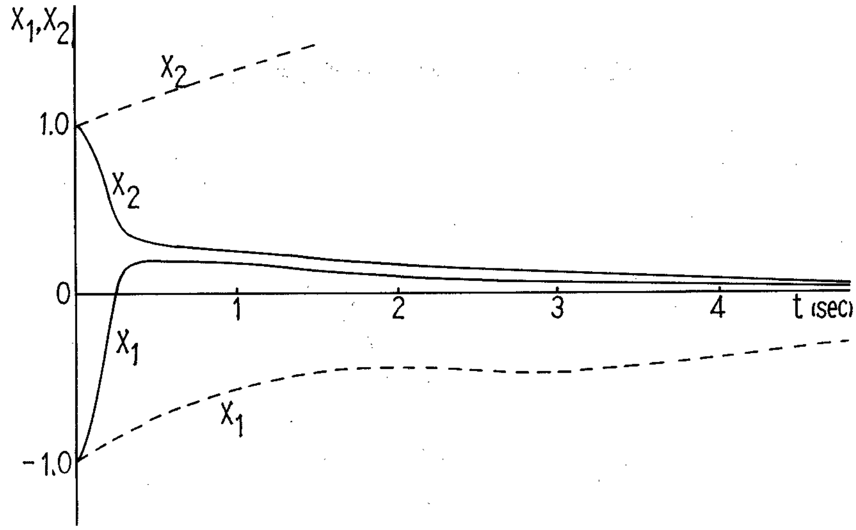

Fig. 2 Time response curves of x_1 and x_2 (Dotted lines are for no control, solid lines are for optimal control.).

case of the optimal nonlinear feedback control the both converge to the origin.

Figure 2 shows the time responses of the states x_1 and x_2 for two cases respectively. Both the x_1 and x_2 decrease in value and approach coincidence and zero without oscillations.

Figure 3 is the same as Fig. 2 but for u_1 and u_2. It is the characteristic of optimal regulator to take the large control values in the initial time.

Figures 4, 5 and 6 illustrate the dominating influence of the initial state (1.0,

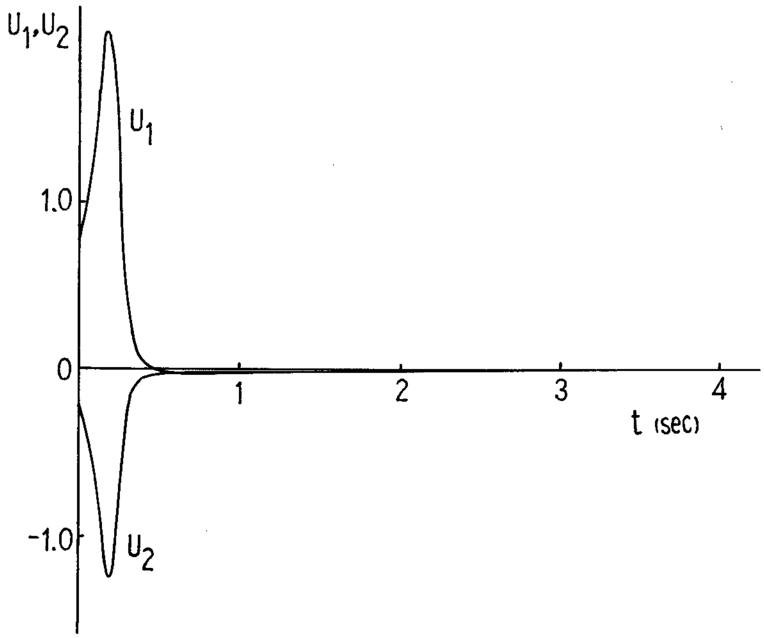

Fig. 3 Time response curves of u_1 and u_2.

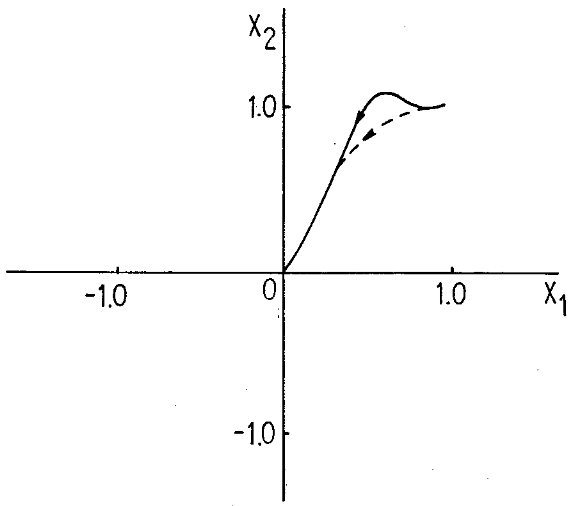

Fig. 4 Phase diagram of x_1 and x_2 (Dotted line is for no control, solid line is for optimal control.).

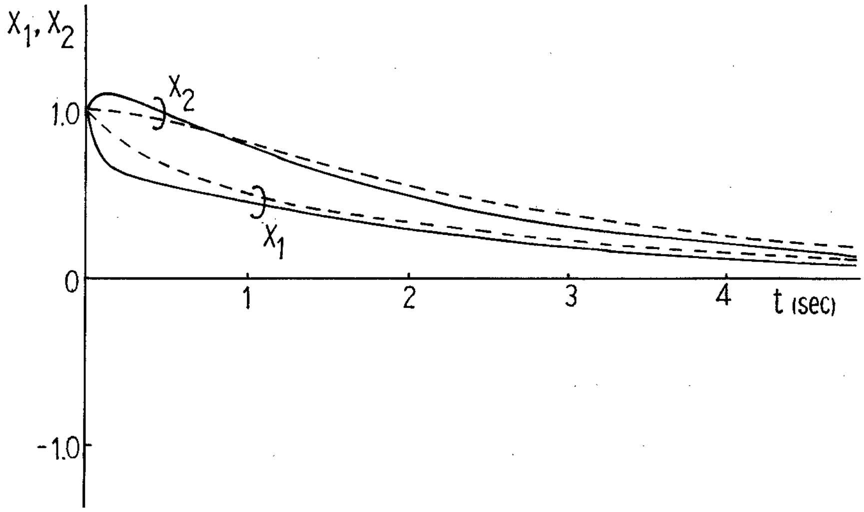

Fig. 5 Time response curves of x_1 and x_2 (Dotted lines are for no control, solid lines are for optimal control.).

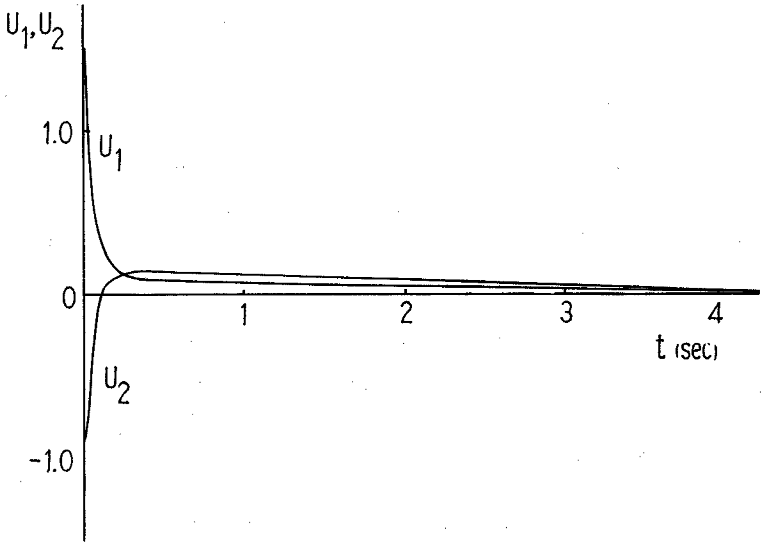

Fig. 6 Time response curves of u_1 and u_2.

1.0). As the time increases, the shapes of curves approach coincidence and zero, and there is little difference of their characteristics between no control and optimal control.

Hence, it will be appreciated that the success of this method depends on the system states being of widely different values.

(Example 2)

Next, we consider the nonlinear system, in Eq. (15),

$$f(x) = \begin{bmatrix} -\sin(x_1 - x_2)\cos(x_1 - x_2) - x_1 \\ \sin(x_1 - x_2)\cos(x_1 - x_2) - x_2 \end{bmatrix} \tag{20}$$

As the Liapunov function, we choose

$$V(x) = \frac{1}{2}\sin^2(x_1 - x_2) + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{21}$$

When $p = 3, k = 6$, the optimal control and dV/dt are given as follows;

$$u = \begin{bmatrix} -(3\sin(x_1 - x_2)\cos(x_1 - x_2) + 2x_1 - x_2)^{\frac{13}{7}} \\ -(-2\sin(x_1 - x_2)\cos(x_1 - x_2) - x_1 + x_2)^{\frac{13}{7}} \end{bmatrix} \tag{22}$$

$$\begin{aligned} \dot{V}(x) = &- [\sin(x_1 - x_2)\cos(x_1 - x_2) + x_1]^2 \\ &- [-\sin(x_1 - x_2)\cos(x_1 - x_2) + x_2]^2 \\ &- [3\sin(x_1 - x_2)\cos(x_1 - x_2) + 2x_1 - x_2]^{\frac{20}{7}} \\ &- [-2\sin(x_1 - x_2)\cos(x_1 - x_2) - x_1 + x_2]^{\frac{20}{7}} \leqslant 0 \end{aligned} \tag{23}$$

In this case, the optimal control Eq. (22) minimizes the following function.

$$\begin{aligned} J = \int_0^\infty [\{\frac{3}{2}\sin 2(x_1 - x_2) + 2x_1 - x_2\}^{\frac{20}{7}} + \{-\sin 2(x_1 - x_2) \\ - x_1 + x_2\}^{\frac{20}{7}}]\, dt \end{aligned} \tag{24}$$

In a similar way, when $p = 3, k = 1$, the following equation are obtained.

$$u = \begin{bmatrix} -(3\sin(x_1 - x_2)\cos(x_1 - x_2) + 2x_1 - x_2)^{\frac{3}{7}} \\ -(-2\sin(x_1 - x_2)\cos(x_1 - x_2) - x_1 + x_2)^{\frac{3}{7}} \end{bmatrix} \tag{25}$$

$$\begin{aligned} \dot{V}(x) = &- [\sin(x_1 - x_2)\cos(x_1 - x_2) + x_1]^2 \\ &- [-\sin(x_1 - x_2)\cos(x_1 - x_2) + x_2]^2 \\ &- [3\sin(x_1 - x_2)\cos(x_1 - x_2) + 2x_1 - x_2]^{\frac{10}{7}} \\ &- [-2\sin(x_1 - x_2)\cos(x_1 - x_2) - x_1 + x_2]^{\frac{10}{7}} \leqslant 0 \end{aligned} \tag{26}$$

$$J = \int_0^\infty [\{\frac{3}{2}\sin 2(x_1 - x_2) + 2x_1 - x_2\}^{\frac{10}{7}} + \{-\sin 2(x_1 - x_2) - x_1 + x_2\}^{\frac{10}{7}}]\, dt \tag{27}$$

Figure 7 illustrates the pase diagram of x_1 and x_2 for two cases of $p = 3$, $k = 6$ and $p = 3$, $k = 1$. Though the x_1 and x_2 converge to the origin in either case, a little difference of their characteristics of convergency can be seen.

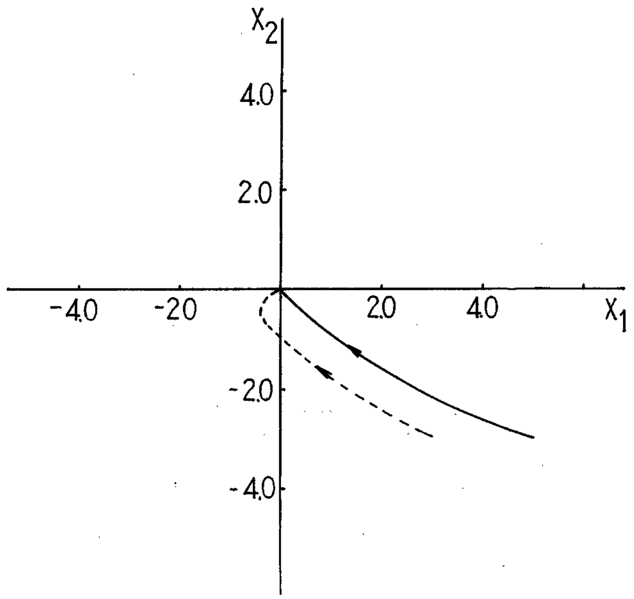

Fig. 7 Phase diagram of x_1 and x_2 (Dotted line is for the case of $p = 3, k = 6$, solid line is for the case of $p = 3, k = 1$.).

Figures 8 and 9 show the time responses of the states x_1, x_2 and the control variables u_1 and u_2. In the case of p=3, k=6, both the deviations of x_1 and x_2 decrease in value rapidly in the early time, but it takes sufficient time until the ultimate convergence to the origin. In the case of p=3, k=1, the deviations of x_1 and x_2 moderately converage to zero. These differences of control characteristics can be deduced from the time response characteristics of u_1 and u_2 in Fig. 9, that is, the time responses of control variables u_1 and u_2 are largely different in the case of $p = 3$, $k = 6$ and in the case of $p = 3$, $k = 1$. In the case of $p = 3$, $k = 6$, seeing the optimal control function Eq. (22), it is obvious that the absolute values of u_1 and u_2 rapidly decrease according to the decrease of x_1 and x_2. As for the case of $p = 3$, $k = 1$, the optimal control variables u_1 and u_2 have reasonable values, even when the states x_1 and x_2 have approached zero. Thus the optimal control in the case of $p = 3, k = 1$ shows better characteristics of convergence in the neighbor of zero. On the contrary, the optimal control in the case of $p = 3$, $k = 6$ is effective for the large deviations of the states x_1 and x_2. Therefore it is to be desired that the suitable values of p and k that meets the purpose of control are chosen.

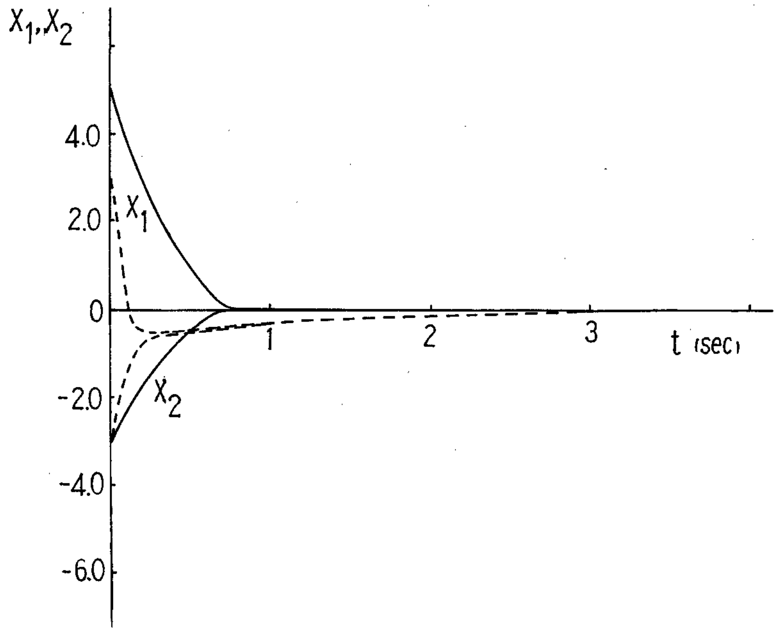

Fig. 8 Time response curves of x_1 and x_2 (Dotted lines are for the case of $p = 3, k = 6$, solid lines are for the case of $p = 3, k = 1$.).

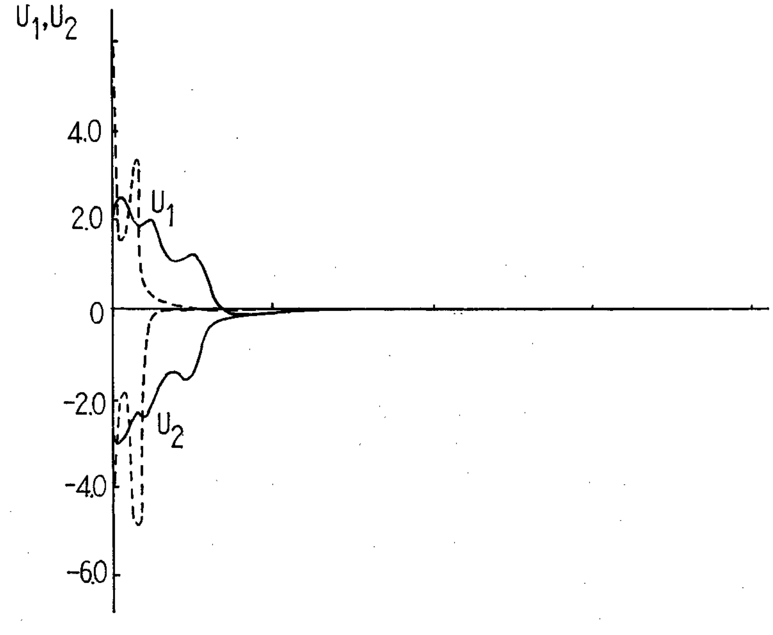

Fig. 9 Time response curves of u_1 and u_2 (Dotted lines are for the case of $p = 3, k = 6$, solid lines are for the case of $p = 3, k = 1$.).

(Example 3)

It is generally difficult to determine $V(x)$ that contents the conditions, $V(x) > 0$, $\dot{V}(x) \leqslant 0$, $V_x(x)f(x) \leqslant 0$.

We next consider, as such example, the system that $f(x)$ is given the following equation in Eq. (15).

$$f(x) = (x_1^2 + x_2, -x_1 + x_2^2)^T \tag{28}$$

In this case, we choose the next equation for $V(x)$.

$$V(x) = x_1^4 + x_2^4 \tag{29}$$

Repeating the procedure used for the previous example, when $p = 1, k = 1$

$$u = \begin{bmatrix} -8x_1^3 + 4x_2^3 \\ 4x_1^3 - 4x_2^3 \end{bmatrix} \tag{30}$$

The optimal control Eq. (30) minimizes the following cost function

$$J = \int_0^\infty [16(5x_1^6 - 6x_1^3x_2^3 + 2x_2^6) + 4x_1x_2(x_2^2 - x_1^2) - 4(x_1^5 + x_2^5)]\, dt \tag{31}$$

In Fig. 10, 11 and 12 are shown the phase diagram of x_1, x_2 and their time responses.

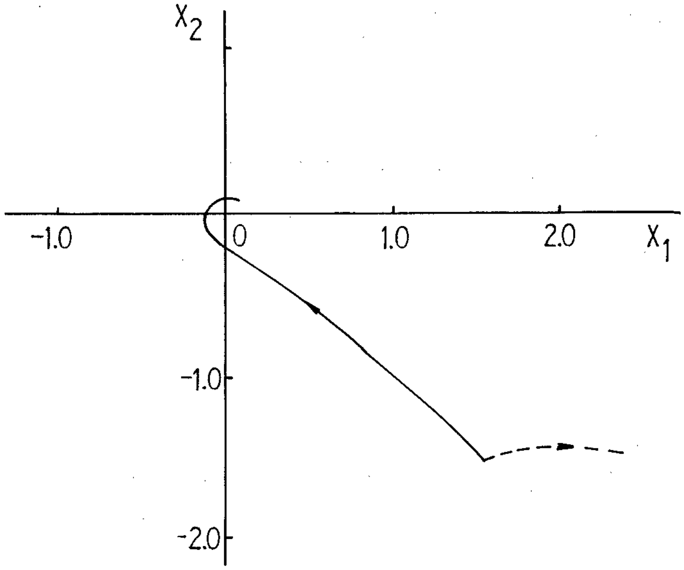

Fig. 10 Phase diagram of x_1 and x_2 (Dotted line is for no control, solid line is for optimal control.).

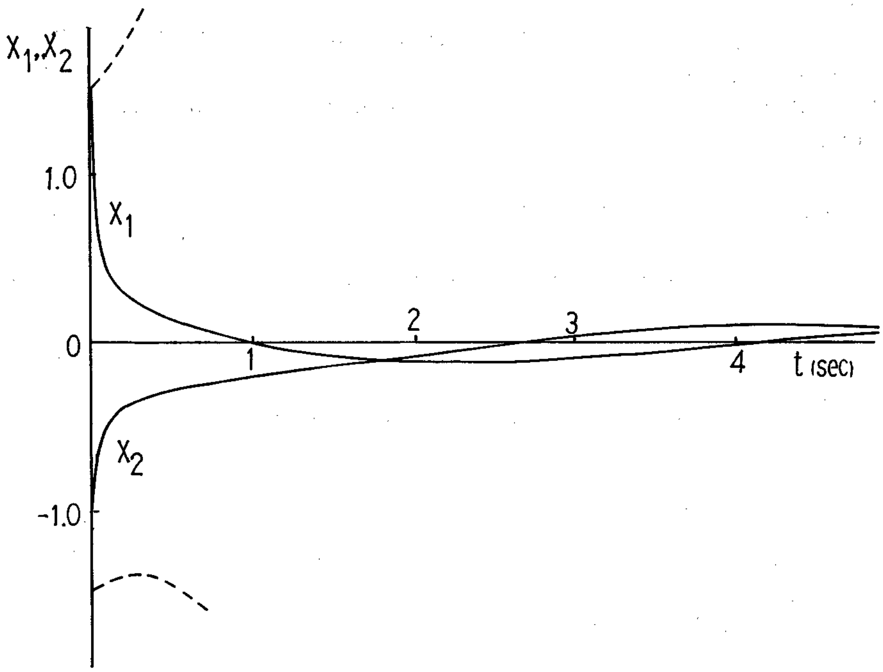

Fig. 11 Time response curves of x_1 and x_2 (Dotted lines are for no control, solid lines are for optimal control.).

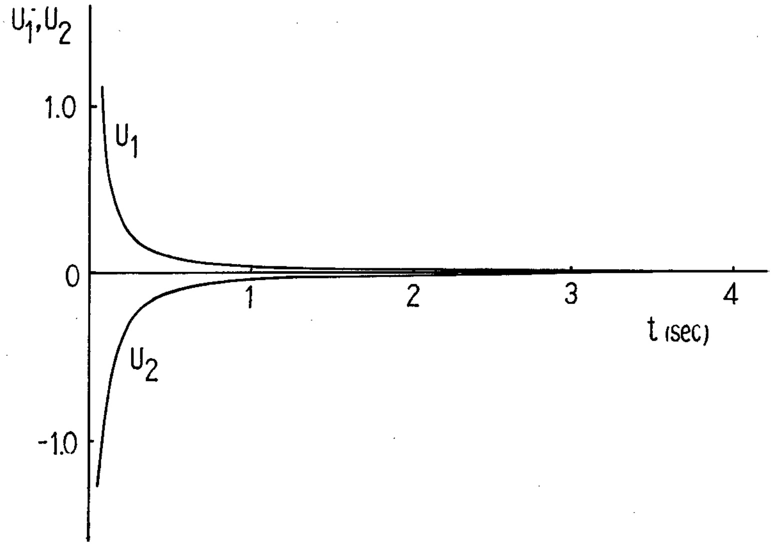

Fig. 12 Time response curves of u_1 and u_2.

In the examples 1 and 2 we were able to find a well-conditioned Liapunov function to derive the nonlinear feedback control for the problem. Unfortunately, equation (29) for example 3 is not always $\dot{V} \leqslant 0$, and so it is difficult to use it in showing that this is the optimum system as were able to do for examples 1 and 2. However, it has been shown that in the domain in which the state deviations are large, the control using it drives the system states to a neighbor of the origin. It is apparent that the examples 1 and 2 are superior to example 3 in terms of response time for a nonlinear system control.

It must be emphasized that the concept of this paper is the direction in which we find improvements in the response time of the system. We comment that these improvements come about basically due to the new optimal function which demands upon our resource.

5. Conclusions

We have shown that certain nonlinear systems can be controlled by using the nonlinear state feedback control. In this paper the method is limited to the nonlinear differential equation systems which are nonlinear with respect to the state variables. However this type of system is seen sometimes in the generator equations in power system. Hence, although the system considered as the numerical examples in this paper was not ones in power system, it is expected that the method proposed here will be useful to control the transient behaviour of the generator in power system.

References

1) F.E. Thau, IEEE Trans., Automatic Control, AC-12, 674 (1967).
2) S.P. Panda, IEEE Trans., Automatic Control, AC-16, 509 (1971).
3) F.E. Thau, IEEE Trans., Automatic Control, AC-16, 511 (1971).
4) O.J. Oaks, Jr. and G. Cook, IEEE Trans., Industrial Erectoronics and Control Instrumentation, IECI-23, 56 (1976).