On the Fourier coefficients of Eisenstein series of Nebentype for Γ0(N)

On the Fourier Coefficients of Eisenstein Series of Nebentype for $\Gamma_0(N)$

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The Fourier coefficients of Eisenstein series of Nebentype for the congruence subgroups of level N are given explicitly.

1. Introduction

Let Z denote a ring of rational integers and $SL_2(Z)$ denote the elliptic modular group defined by

$$SL_2(Z)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad-bc=1, a, b, c, d \in Z\right\}.$$

Let $\Gamma(N)$ denote the principal congruence subgroup (of $SL_2(Z)$) of level N, i.e.,

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \in SL_2(Z) \;\middle|\; \left(a \equiv d \equiv 1, b \equiv c \equiv 0 \; mod\, N\right)\right\}.$$

We denote by $\Gamma_0(N)$ the congruence subgroup of level N defined as follow:

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \;\middle|\; c \equiv 0 \; mod\, N\right\}.$$

Let C denote the complex field, and H denote the complex upperhalf plane i.e.

$$H=\left\{z \in C \;\middle|\; Im(z)>0\right\}.$$

For every integer k, $L \in SL_2(Z)$, and a function $f(\tau)$ on H, we write

$$f(\tau)\,|_k L=f\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-k}.$$

Let k be an integer. A C-valued function $f(\tau)$ on H is called a modulor form of weight k with respect to $\Gamma(N)$, if $f(\tau)$ satisfies the following three conditions:

(i) $f(\tau)$ is holomorphic on H;
(ii) $f(\tau)\,|_k L=f(\tau)$ for all $L \in \Gamma(N)$;
(iii) $f(\tau)$ is holomorphic at every cusp of $\Gamma(N)$.

Let $N=p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ be the prime decomposition of N. Then we define the quadratic Dirichlet character χ mod N by

$$\chi(n)=\chi_N(n)=\left(\frac{n}{p_1}\right)^{r_1}\left(\frac{n}{p_2}\right)^{r_2} \cdots\cdots\left(\frac{n}{p_s}\right)^{r_s} \quad \text{for all } n \in Z,$$

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where $(-)$ denotes the Legendre symbol.

Let $F(\tau)$ be a modular form of weight k with respect to $\Gamma(N)$. If $F(\tau)$ satisfies the additional condition:

$$F(\tau)\mid_k L = \chi(d)F(\tau) \text{ for all } L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N),$$

then $F(\tau)$ is called a modular form of (Neben) type $(-k, N, \chi)$. The aim of the present paper is to constract a complete system of Eisenstein series of type $(-k, N, \chi)$ and to give their Fourier coefficients explicitely. It is remarked that the proof of Theorem announced in [2] is contained in this note.

2. Eisenstein Series of Type $(-k, N, \chi)$

Let the notation be as in the preceeding sections. Hereafter we assume $\chi(-1) = (-1)^k$. This assumption is necessary to assure the existence of non-zero modular forms of type $(-k, N, \chi)$.

First of all we shall review some results for the Eisenstein series $G_k(\tau; a_1, a_2, N)$ of the group $\Gamma(N)$ (Hecke [5]).

Let a_1, a_2 and k be rational integers. If $k > 2$, then the series $G_k(\tau; a_1, a_2, N)$ is a function on H defined as follows.

$$(1) \quad G_k(\tau; a_1, a_2, N) = \sum_{\substack{m_1 \equiv a_1 \ (\mathrm{mod}\ N) \\ m_2 \equiv a_2 \ (\mathrm{mod}\ N)}}{}' (m_1\tau + m_2)^{-k},$$

where τ denotes a variable on H and the summation Σ' runs through the pair of integers such that $(m_1, m_2) \equiv (a_1, a_2) \bmod N$, $(m_1, m_2) \neq (0, 0)$.

It is well known that their Fourier expansions at the cusp $i\infty$ of $\Gamma(N)$ is given as follows.

$$(2) \quad G_k(\tau; a_1, a_2, N) = \delta(a_1/N) \sum_{m_2 \equiv a_2 (N)} 1/m^k$$
$$+ (-2\pi i)^k N^{-k}/(k-1)! \sum_{\substack{mm_1 > 0 \\ m_1 \equiv a_1 (N)}} m^{k-1} sgn(m)\, \xi_N^{a_1 m} \exp(2\pi i m m_1 \tau/N),$$

where

$$\delta(x) = \begin{cases} 1 & \text{if } x \text{ is a rational integer} \\ 0 & \text{oterwise,} \end{cases}$$

$$\xi_N = \exp(2\pi i/N),$$

and

$$sgn(x) = x/|x| \text{ for } x \neq 0;\ sgn(0) = 0.$$

If $k = 2$, there we define $G_k(\tau, a_1, a_2, N)$ by the following Fourier series:

$$(3) \quad G_k(\tau; a_1, a_2, N) = -2\pi i N^{-2}/(\tau - \bar{\tau})$$

$$+ \delta(a_1/N) \sum_{m_2 \equiv a_2 (N)} 1/m^2 - 4\pi^2/N^2 \sum_{\substack{mm_1>0 \\ m_1 \equiv a_1 (N)}} |m| \xi_N^{a_1 m} \exp(2\pi i m m_1 \tau/N).$$

(See p. 469 of [5] for the original definition of $G_2(\tau, a_1, a_2, N)$.)

For every $k \geqslant 2$, we know that the series $G_k(\tau; a_1, a_2, N)$ satisfy the following properties (4) and (5).

(4) $G_k(\tau; a_1, a_2, N) = G_k(\tau; b_1, b_2, N)$, if $a_1 \equiv b_1$ and $a_2 \equiv b_2$ $(mod\, N)$.

(5) $G_k(\tau; a_1, a_2, N) \mid_k L = G_k(\tau; aa_1 + ca_2, ba_1 + da_2, N)$, for each $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(Z)$.

Let $G_k(\tau; a_1, a_2, N)$ denote as follows with the integers a_1, a_2 such that $(a_1, a_2, N) = 1$:

$$G_\kappa^*(\tau; a_1, a_2, N) = \sum_{\substack{m_i \equiv a_i (N) \\ (m_1, m_2) = 1}} (m_1\tau + m_2)^{-k}.$$

Then it is well known that $G_\kappa^*(\tau; a_1, a_2, N)$ is not zero at parabolic vertex $-a_2/a_1$ and zero at otherwise. And the relation between $G_k(\tau; a_1, a_2, N)$ and $G_k^*(\tau; a_1, a_2, N)$ is as follows:

$$(6) \quad G_k(\tau; a_1, a_2, N) = \sum_{t \bmod N} G_\kappa^*(\tau; ta_1, ta_2, N) \sum_{tn \equiv 1 (N)} 1/n$$

$$(7) \quad G_\kappa^*(\tau; a_1, a_2, N) = \sum_{t \bmod N} G_k(\tau; a_1, a_2, N) c_t,$$

$$c_t = \sum_{\substack{tn \equiv 1 (N) \\ n>0}} \mu(n)/n^k,$$

where $\mu(n)$ is Mobiuss function.

Lemma 1. *Let Γ_∞ denote the isotropy subgroup of $\Gamma(1)$ at the cusp $i\infty$. Then cardinality of $\Gamma_0(N)\backslash\Gamma(1)/\Gamma_\infty$ is equal to $\sum_{t|N} \varphi((t, N/t))$ (here we denote by φ the Euler functon), and also is equal to the number of iquivalent classes of cusps of $\Gamma_0(N)$. Moreover each class of cusps $\Gamma_0(N)$ is represented a pair of coprime integers $\{x, r\}$, where r is a positive divisor of N and x is uniquely determined mod $(r, N/r)$.*

Proof. We refer to [1] or [3] for the proof of this lemma.

Q.E.D.

For each class of cusps represented by the pair$\{x, r\}$let us define the Eisenstein series $E_{\{x,r\}}^*(\tau)$ as follows:

$$(8) \quad E_{\{x,r\}}^*(\tau) = \sum_{\substack{a \bmod N \\ b \bmod N}} \chi(a) G_k(\tau; xra, xrb + a', N),$$

where a' is an integer uniquely determined $mod\, N$ such that $aa' \equiv 1$ (N).

Theorem 1. *Let $E_{\{x,r\}}^*(\tau)$ be the series defined as above. Then the following state-*

ments hold:

(i) $E^*_{\{x,r\}}(\tau)$ *are modular form of type* $(-k, N, \chi)$.

(ii) Every modular form of type $(-k, N, \chi)$ *is expressed as linear combination of* $E^*_{\{x,r\}}(\tau)$ *mod cusps forms.*

Proof. For the class $\{x, r\}$, it is well known that x is chosen in such a way $0<x<N$, $(x, N)=1$ and that the class $\{x, r\}$ contains the cusp $-1/xr$ (see [1]).

Let L be a element of $\Gamma_0(N)$ such that $L=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. Then by (4) and (5), we have the following:

(9) $G_k(\tau; xr, 1, N) \mid_k L = G_k(\tau; xra, xrb+d, N)$.

Put $E^{**}_{\{x,r\}}(\tau) = \sum_{\substack{a \bmod N \\ b \bmod N}} \chi(a) G^*_\kappa(\tau; xra, xrb+a', N)$.

Then it is easy to see that $E^*_{\{x,r\}}(\tau)$ is not zero at any cusps in $\{x, r\}$ and zero at other cusps. Futher $E^*_{\{x,r\}}(\tau)$ and $E^{**}_{\{x,r\}}(\tau)$ satisfy the similar linear relations to (6) and (7).

Invew of (5), (8) and (9), it is easy to see that $E^*_{\{x,r\}}(\tau)$ are modular form of *type* $(-k, N, \chi)$ of weight k for $\Gamma_0(N)$.

Let $f(\tau)$ be an arbitrary modular form of type $(-k, N, \chi)$ for $\Gamma_0(N)$ with value $a_{\{x,r\}}$ at the cusp $-1/xr$, and $e_{\{x,r\}}$ be value of $E^*_{\{x,r\}}(\tau)$ at the cusp $-1/xr$. Then

$$f(\tau) - \sum_{\{x,r\}} a_{\{x,r\}}/e_{\{x,r\}} E^*_{\{x,r\}}(\tau)$$

is a cusp form of type $(-k, N, \chi)$ for $\Gamma_0(N)$. Q.E.D.

3. The Fourier Coefficients of $E_{\{l,t\}}(\tau)$

We consider the normalised Eisenstein series $E_{\{l,t\}}(\tau)$ in place of $E^*_{\{l,t\}}(\tau)$. Then by (2) or (3) $E_{\{l,t\}}(\tau)$ are given as follows.

$$2(-2\pi i)^k E_{\{1,1\}}(\tau) = (k-1)! \sum_{\substack{a \bmod N \\ b \bmod N}} \chi(a) G_k(\tau; a, b, N),$$

$$2(2\pi)^k E_{\{1,N\}}(\tau) = \gamma_k (k-1)!\ N^{k-1/2} \sum_{a \bmod N} \chi(a) G_k(\tau; 0, a, N),$$

and for $\{x, r\}$ other than $\{1, l\}$ and $\{1, N\}$,

$$2(-2\pi i)^k (r, N/r) E_{\{x,r\}}(\tau) = r^{k-1}(k-1)!\ \chi_{N/r}(x)$$

$$\times \sum_{\substack{a \bmod N \\ b \bmod N}} \chi(a) G_k\ (\tau; xra, xrb+a', N).$$

Put $r_1 = r/(r, N/r)$ and define $A_k(m, a)$ as follows:

$$A_k(m, a) \sum_{i \bmod r_1} \chi_r(a+Ni/r)\, \xi^{(a+Ni/r)m}.$$

Then it is easy to see that the following (10) are formed:

(10) $A_r(m, a) = A_r(m, b)$ if $a \equiv b \bmod N/r$.

Theorem 2. *Keep the notations as above. Then the Fourier expantions with respect to $z = exp(2\pi i)$ of $E_{\{x,r\}}(\tau)$ are given as follows.*

$$E_{\{1,1\}}(\tau) = \sum_{n=1}^{\infty} c_0(n) z^n,\ c_0(n) = \sum_{d|n} d^{k-1}\chi(n/d);$$

$$E_{\{1,N\}}(\tau) = c_\infty(0) + \sum_{n=1}^{\infty} c_\infty(n) z^n,$$

$$C_\infty(N) = \begin{cases} \sum_{d|n} d^{k-1}\chi(d) & if\ n=0, \\ \gamma_k N^{k-1/2}(k-1)!\ (2\pi)^{-k} L(k,\chi) & if\ n=0, \end{cases}$$

where $L(k,\chi) = \sum_{n=1}^{\infty} \chi(n)\, l/n^k$.

And for $\{x, r\}$ other than $\{1, 1\}$ and $\{1, N\}$

$$E_{\{x,r\}}(\tau) = \sum_{n=1}^{\infty} c_{\{x,r\}}(n) z^n,$$

$$c_{\{x,r\}}(n) = \sum_{d|n} d^{k-1}\chi_{N/r}(n/d) A_r(d, nx'/d).$$

Proof. For the Fourier coefficients of $E_{\{1,1\}}(\tau)$ and $E_{\{1,N\}}(\tau)$, we refer to [2]. For the purpose of expanding $E_{\{x,r\}}(\tau)$, we start from giving the following obvious equalitys (11) and (12):

$$(11)\quad \sum_{b \bmod N} \xi_{N/r}^{bm} = \begin{cases} N & \text{if } m \text{ is divided by } N/r, \\ 0 & \text{otherwise,} \end{cases}$$

$$(12)\quad \sum_{c \equiv a(N/r)} \chi_r(c)\xi_r^{cm} = (r, N/r) A_r(m, a).$$

Next, by (2) or (3) we have

$$N^k(k-1)!\ E^*_{\{x,r\}}(\tau) = N^k(k-1)! \sum_{\substack{a \bmod N \\ b \bmod N}} \chi(a) G_k(\tau; rxa, rxb + a', N)$$

$$= (-2\pi i)^k \sum_{\substack{a \bmod N \\ b \bmod N}} \chi(a) \sum_{\substack{mm_1>0 \\ m_1 \equiv xra(N)}} m^{k-1} sgn(m)\, \xi_N^{(rxb+a')m} \exp(2\pi i m m_1 \tau/N)$$

$$= (-2\pi i)^k \Big(\sum_{b \bmod N} \xi_{N/r}^{xbm} \Big) \sum_{a \bmod N} \chi(a) \sum_{\substack{mm_1>0 \\ m_1 \equiv xra(N)}} m^{k-1} sgn(m)\, \xi_N^{a'm}$$

$$\times \exp(2\pi i\, m m_1 \tau/N).$$

By (11)

$$r^{k-1}(k-1)!\ E_{\{x,r\}}(\tau)$$

$$= (-2\pi i)^k \sum_{a \bmod N} \chi(a) \sum_{\substack{mm_1>0 \\ m_1 \equiv xra(N)}} m^{k-1} sgn(m)\, \xi_r^{a'm} \exp(2\pi i m m_1 \tau/r)$$

$$= (-2\pi i)^k \sum_{a \bmod N/r} \chi_{N/r}(a) \sum_{\substack{mm_1 > 0 \\ m_1 \equiv xra(N)}} \Big(\sum_{c \equiv a(N/r)} \chi_r(c)\, \xi_r^{cm} \Big) m^{k-1} sgn(m)$$

$$\times \exp(2\pi i m m_1 \tau/r) .$$

By (12)

$$r^{k-1}(k-1)!\ E_{\{x,r\}}(\tau) = (-2\pi i)^k (r, N/r) \sum_{a \bmod N/r} \chi_{N/r}(a)$$

$$\times \sum_{\substack{mm_1 > 0 \\ m_1 \equiv xra(N)}} m^{k-1} sgn(m) \sum_{i \bmod r_1} \chi_r(a + Ni/r)\, \xi_r^{(a+Ni/r)m} \exp(2\pi i m m_1 \tau/r)$$

$$= (-2\pi i)(r, N/r) \sum_{a \bmod N/r} \chi_{N/r}(a) \sum_{\substack{mm_1 > 0 \\ m_1 \equiv xra(N)}} m^{k-1} sgn(m) A_r(m, a)$$

$$\times \exp(2\pi i m m_1 \tau/r)$$

$$= 2(-2\pi i)(r, N/r) \sum_{a \bmod N/r} \chi_{N/r}(a) \sum_{\substack{m_1 \equiv xa(N/r) \\ m_1 > 0}} m^{k-1} A_r(m, a) \exp(2\pi i m m_1 \tau) .$$

Put $z = \exp(2\pi i)$, then by (10) we have

$$r^{k-1}(k-1)!\ E_{\{x,r\}}(\tau)$$

$$= 2(-2\pi i)(r, N/r) \sum_{a \bmod N/r} \sum_{\substack{m_1 \equiv xa(N/r) \\ m_1 > 0}} m^{k-1} \chi_{N/r}(x' m_1) A(m, x'm)$$

$$= 2(-2\pi i)(r, N/r) \chi_{N/r}(x) \sum_{n=1}^{\infty} \sum_{d \mid n} d^{k-1} \chi_{N/r}(n/r) A_r(d, nx'/d) z^n$$

$$= 2(-2\pi i)(r, N/r) \chi_{N/r}(x) \sum_{n=1}^{\infty} c_{\{x,r\}}(n) z^n .$$

Q.E.D.

Corollary. *Let the notations be as above. Then we have the following relation among* $E_{\{x,r\}}(\tau)$.

$$\sum_{x}{}' E_{\{x,r\}}(\tau) = \tilde{\gamma}_r r^{1/2} \sum_{n=1}^{\infty} \sum_{d \mid n} d^{k-1} \chi_r(d) \chi_{N/r}(n/d) z^n ,$$

where $\tilde{\gamma}_l$ *is defined as follows:*

$$\tilde{\gamma}_l = \begin{cases} 1 & \text{if } l \equiv 1 \bmod 4, \\ i\,(=\sqrt{-1}\,) & \text{if } l \equiv 3 \bmod 4. \end{cases}$$

Proof. Here, we shall give a detailed way in which x is chosen. Put $x = x(r, i)$ with a positive integer i such that $1 \leqslant i \leqslant \varphi((r, N/r))$. Then $x(r, i)$ is chosen as follows (see [1]):

$$x(r, i) = \begin{cases} i & \text{if } (N, i) = 1, \\ j & \text{if } (N, i) \neq 1, \end{cases}$$

where j is a positive integer such that $j \equiv i \bmod r$, $(j, N) = 1$ and $\varphi((r, N/r)) < j < N$.

Let x be as above. Then we have the following equality by applying Gussian Sum and since $r_1 = r/(r, N/r)$.

$$\sum_x c_{\{x,r\}}(n)$$

$$= \sum_{d|n} \chi_{N/r}(n/d) \sum_x \Big(\sum_{i \bmod r_1} \chi_r(nx'/d + Ni/r)\, \xi_r^{(nx'/d+N/r)d} \Big)$$

$$= \sum_{d|n} \chi_{N/r}(n/d) \sum_{c \bmod r} \chi_r(c)\, \xi_r^{cd}$$

$$= \tilde{\gamma}_r \sqrt{r} \sum_{d|n} \chi_r(d)\, \chi_{N/r}(n/d)\,.$$

Q.E.D.

Now, Let r be square, then $\chi_r(*) = 1$. Hence we have

$$\sum_{i \bmod r_1} \chi_r\,(nx'/d + Ni/r)\, \xi_r^{(nx'/d+Ni/r)d} = \sum_{i \bmod r_1} \chi_r\,(n/d + Ni/r)d$$

$$= \xi_r^{nx'} \sum_{i \bmod r_1} \xi_{r_1}^{di}$$

$$= \begin{cases} \varphi(r_1)\, \xi_g^{n_1 x'} & \text{if } r_1 \text{ devides } d, \\ 0 & \text{if otherwise.} \end{cases}$$

where $r_2 = N/r\,(r, N/r)$, $g = (r, N/r)$ and $n_1 = n/r_1$.

Next, if r is square free and $(r, N/r) = 1$, then $x = 1$ and $r_1 = r$. Hence we have the following equality by using Gussian Sum:

$$\sum_{i \bmod r_1} \chi_r(nx'/d + Ni/r)\, \xi_r^{(nx'/d+Ni/r)d} = \sum_{i \bmod r} \chi_r(n/d + Ni/r)\, \xi_r^{(n/d+Ni/r)d}$$

$$= \sum_{c \bmod r} \chi_r(c)\, \xi_r^{cd}$$

$$= \tilde{\gamma}_r \sqrt{r}\, \chi_r(d)\,.$$

Here we have the following proposition.

Proposition 1. *Let the notations be as above. Then if r is square,*

$$A_r(d, nx'/d) = \begin{cases} \varphi(r_1)\, \xi_g^{n_1 x'} & (\textit{if } r_1 \textit{ devides } d), \\ 0 & (\textit{otherwise}), \end{cases}$$

and if r is square free and $(r, N/r) = 1$,

$$A_r(d, nx'/d) = \tilde{\gamma}_r \sqrt{r}\, \chi_r(d)\,.$$

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