Geometry of Catastrophe Model

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The method for drawing the cuspoid (fold, cusp, swallowtail and butterfly) catastrophe manifold and the swallowtail bifurcation set by 3-dimensional space is shown. Moreover, the method for drawing the ruled surface and bifurcation set of elementary catastrophe by 2-dimensional space is established. Based on these results, the geometrical features of cuspoid catastrophe are discussed.

1. Introduction

Catastrophe theory, which was originated by Rene Thom, has been applied to many fields such as behavioral science, psychology, physics and economics, etc[1]. Catastrophe theory describes discontinuous phenomenon under gradually changing situation. However, there exist many problems in the application of catastrophe theory. One of the problems is that it is impossible to grasp exactly the geometrical features of higher-dimentional catastrophe model more than 4-dimension. So far, precise geometry of catastrophe model is represented only by 2-dimensional ruled surface developed by A. E. R. Woodcock and T. Poston[2]. Precise representation of catastrophe model by 3-dimensional space is not realized.

This paper describes the method for drawing the geometry of cuspoid (fold, cusp, swallowtail and butterfly) catastrophe by 3-dimensional space, and their geometrical features are discussed.

2. Geometry of Cuspoid Catastrophe

In this section, the geometry of cuspoid catastrophe is shown, and their features are discussed. Cuspoid catastrophe has one state variable. Fold, cusp, swallowtail and butterfly, etc. are included in this type of catastrophe. The classification of elementary catastrophe is shown in Table 1. Codimension means the number of control variables. Universal unfolding equals to the potential function.

2.1 Cusp Catastrophe

Cusp catastrophe is most frequently applied to many fields, and has the following properties:

1 : The state variable is bimodal for some values of control factors.
2 : Abrupt, catastrophic changes are observed between upper and lower attractors.
3 : There is hysterisis, that is, the abrupt change from one attractor to another which takes place at different values of control factors depending on the direction of change.

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4 : There is an inaccessible zone of behavior for some values of control factors.
5 : There is the possibility of divergent behavior.

Cusp catastrophe manifold is shown in Fig. 1, and is given by the following equation.

$$-x^3 + bx + a = 0 \tag{1}$$

where x is the state variable, and a and b are normal factor and splitting factor, respectively. Dual cusp is taken up here.

Fig. 1 is drawn as follows. First, the value of b-axis is fixed, and the value of x-axis is increased at a constant interval. Then, the corresponding value of a-axis is calculated, and one curve is drawn. The cusp catastrophe manifold is constituted by the set of curves when the value of b is varied. The increment of vertical direction is constant, because the value of x increases at a constant interval. In this case, one value of horizontal direction corresponds to one value of vertical direction. In other words, this is one-to-one mapping. The value X of horizontal direction corresponding to the value Y of vertical direction is memorized as follows.

$$I = Y/\delta \tag{2}$$

$$U(I) = X \tag{3}$$

δ : increment of vertical direction

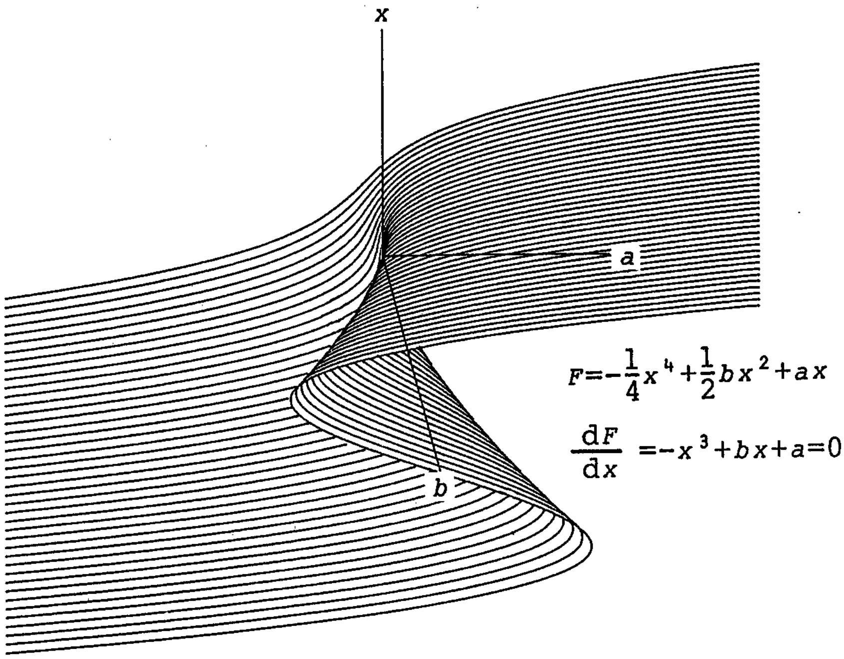


Fig. 1 Cusp catastrophe manifold

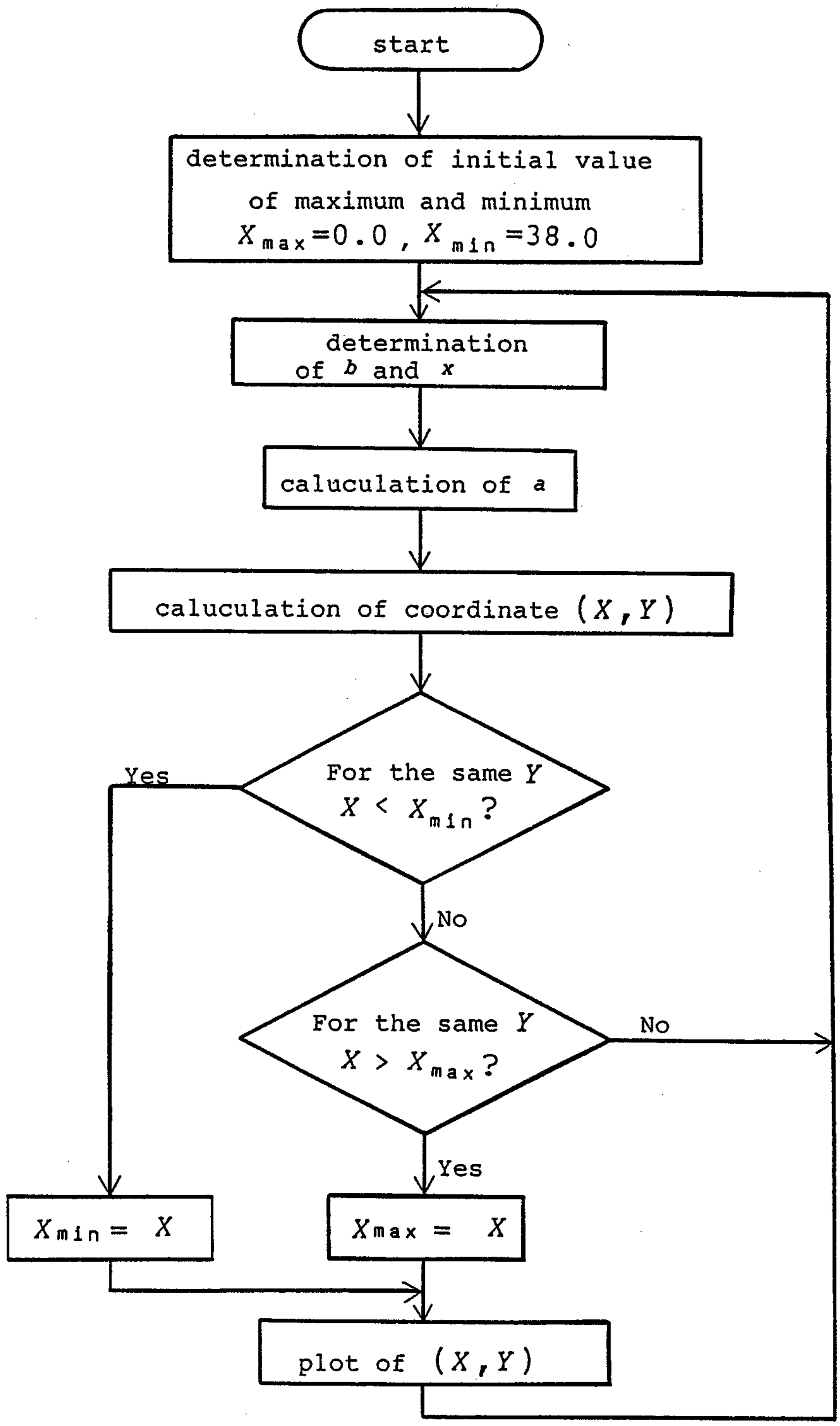


Fig. 2 Flow chart for drawing cusp catastrophe manifold

The calculated point (X, Y) is plotted only when X is greater than X_{max} or less than X_{min}. X_{max} and X_{min} are the maximum value and the minimum value of $U(I)$, respectively. If the point (X, Y) is plotted, $X\{$i e., $U(I)\}$ is memorized as X_{max} or X_{min}. This procedure is summarized in Fig. 2. Swallowtail catastrophe manifold and butterfly catastrophe manifold can be drawn similarly.

2.2 Swallowtail Catastrophe

Fig. 3 and Fig. 4 are the swallowtail catastrophe manifold for $c > 0$ and $c < 0$, respectively. Fig. 5 shows the swallowtail bifurcation set. On the bifurcation set, catastrophic change occurs. The shape of catastrophe manifold and bifurcation set differ for $c > 0$ and $c < 0$.

Fig. 5 cannot be drawn by means of the procedure shown in Fig. 2. The swallowtail bifurcation set is given by

$$\left.\begin{aligned} & x^4 + cx^2 + bx + a = 0 \\ & 4x^3 + 2cx + b = 0 \end{aligned}\right\} \tag{4}$$

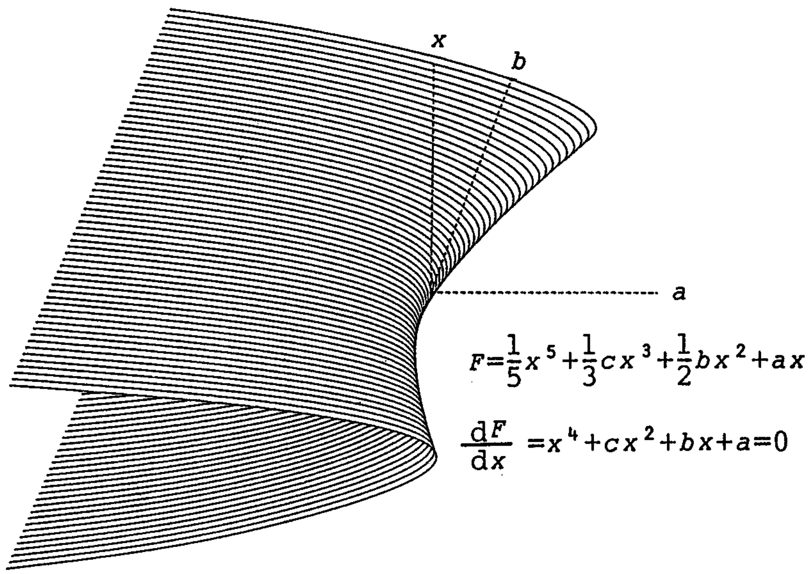


Fig. 3 Swallowtail catastrophe manifold ($c > 0$)

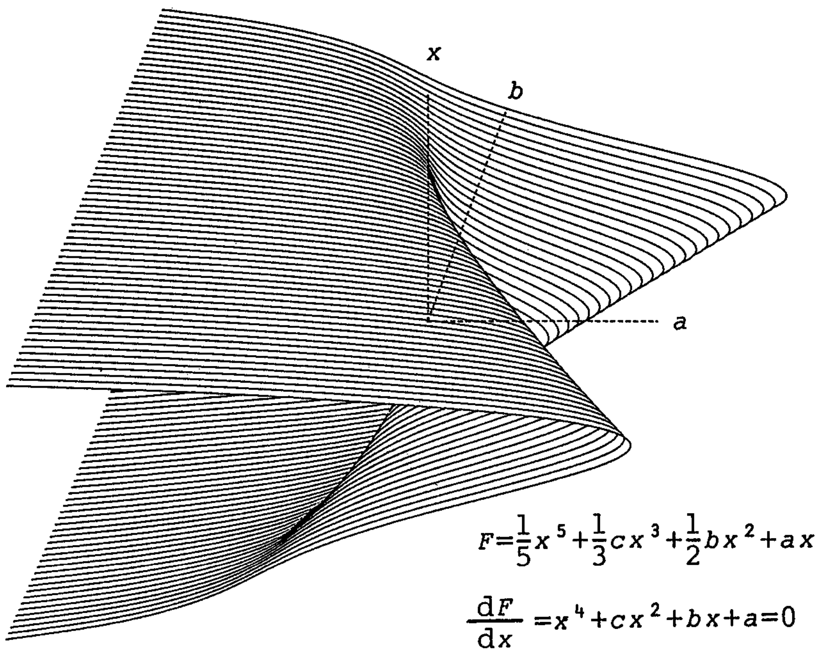


Fig. 4 Swallowtail catastrophe manifold ($c<0$)

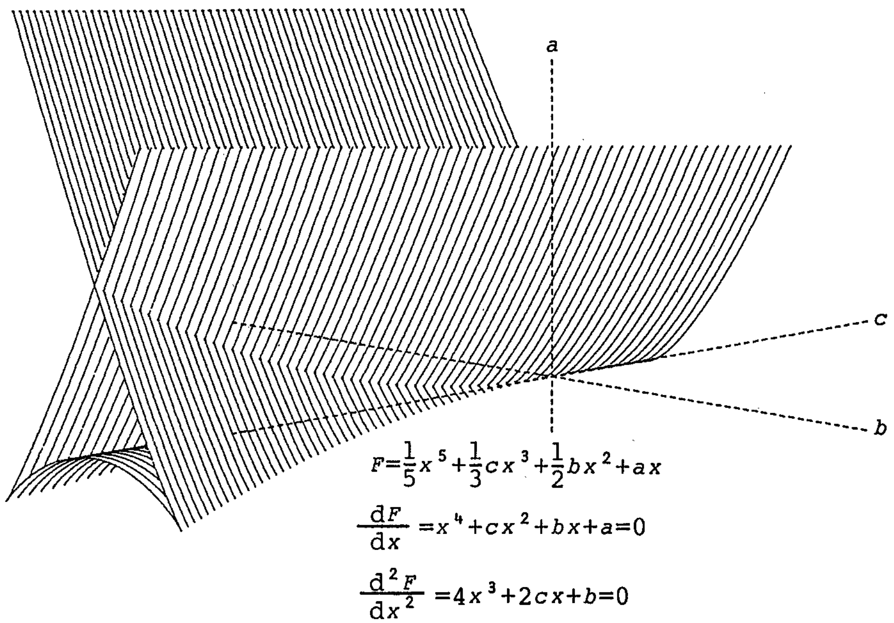


Fig. 5 Swallowtail bifurcation set

Here x is the state variable, and a,b, and c are control variables. The value of c is fixed, and the values of a and b are calculated when the value of x increases at a constant interval. In this case, neither vertical direction to horizontal direction nor horizontal direction to vertical direction follows one-to-one mapping. Accordingly, Fig. 5 is drawn as follows.

The coordinate (X, Y) for the minimum value of c is memorized as the boundary value. One curve for the fixed value of c is divided into three parts so that one value of the horizontal direction corresponds to one value of the vertical direction. (See Fig. 6) For the sake of convenience, the part of solid line shall be called triangular cone. Furthermore the triangular cone is divided into three part, that is, left side part, middle part and right side part. The procedure of Fig. 2 can be applied to each part. However, in this case both values of horizontal direction and vertical direction do not change at a constant interval. So, the linear approximation is used between (X, Y) and $(X'$, $Y')$, and the coordinate of these points between (X, Y) and (X', Y) are calculated. The coordinate (X', Y') is the points calculated one step before. The value of X-coodinate for these points are newly set up as maximum or minimum. In this way, the treatment shown in Fig. 2 is possible. This procedure is summarized in Fig. 7 and Fig.8.

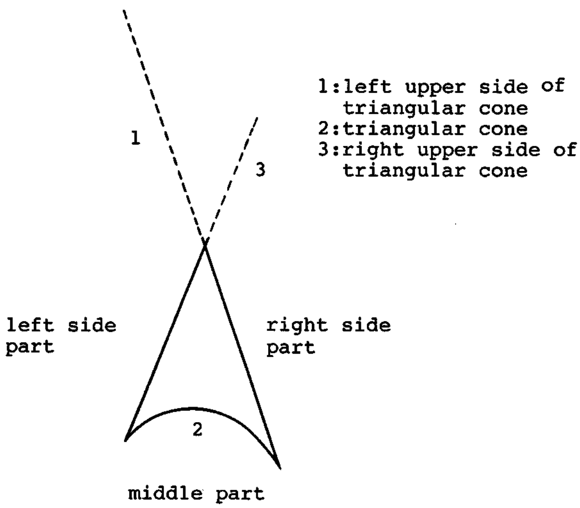


Fig. 6 Name of boundary for drawing the swallowtail bifurcation set

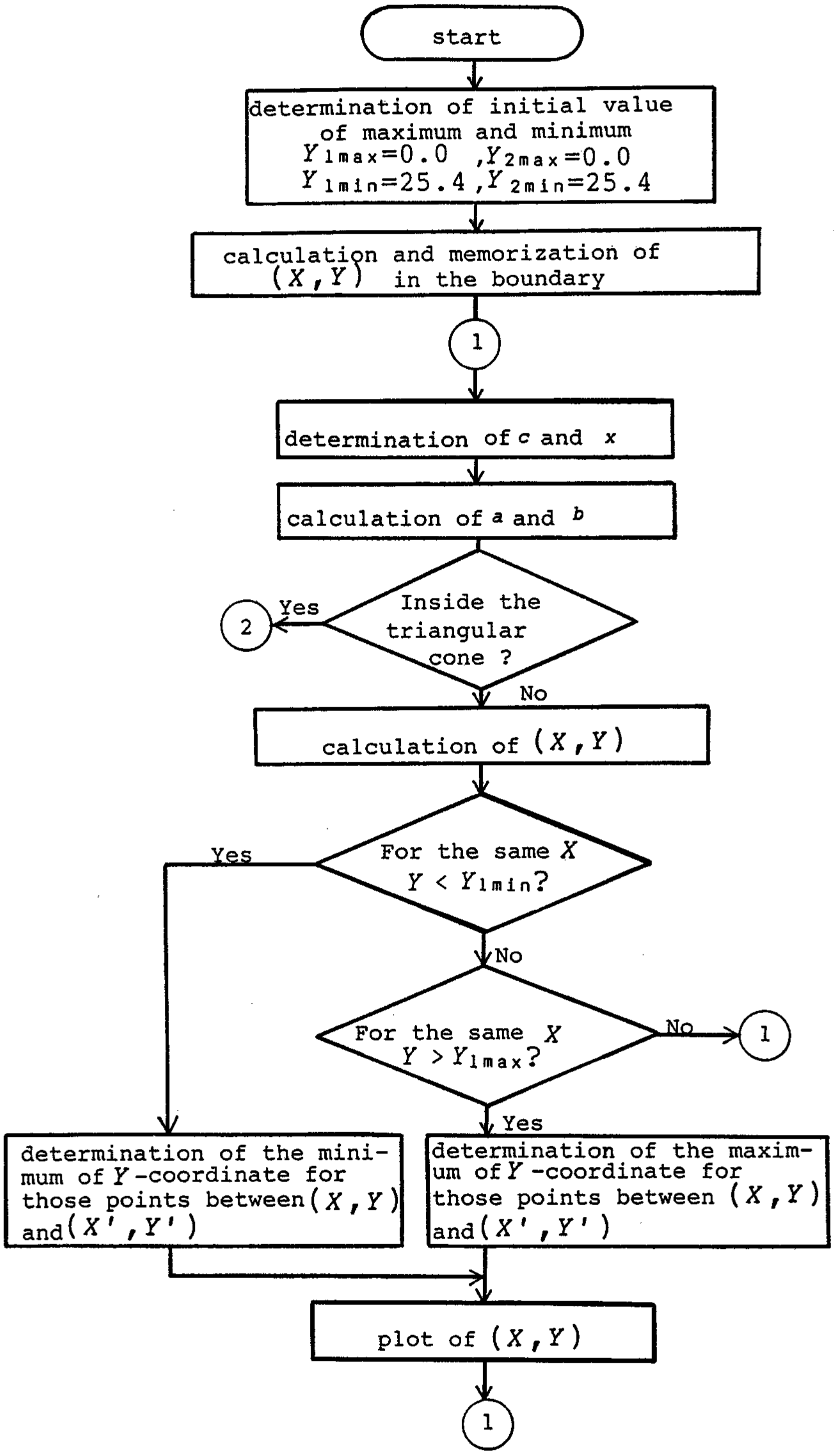


Fig. 7 Flow chart for drawing the swallowtail bifurcation set (1)

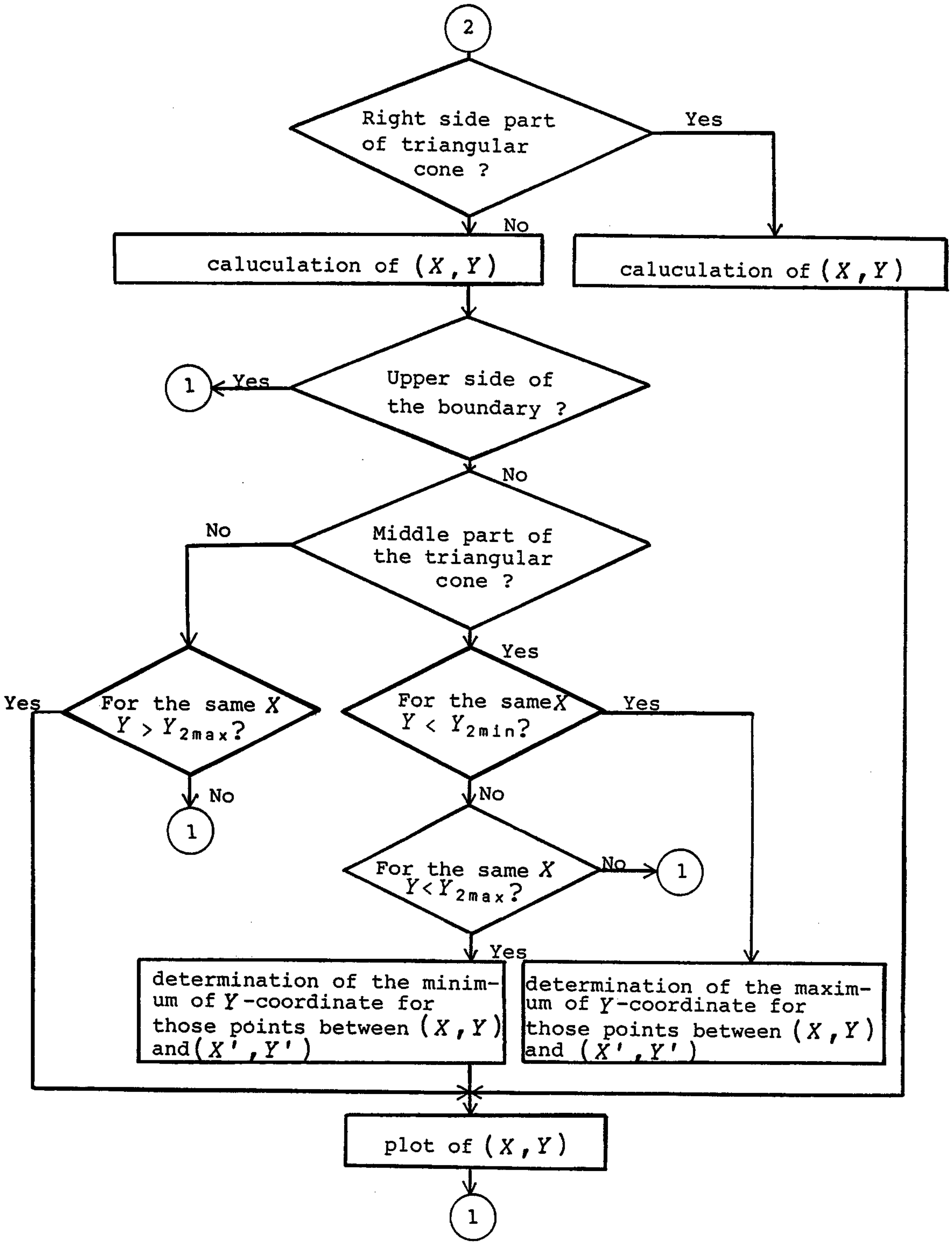
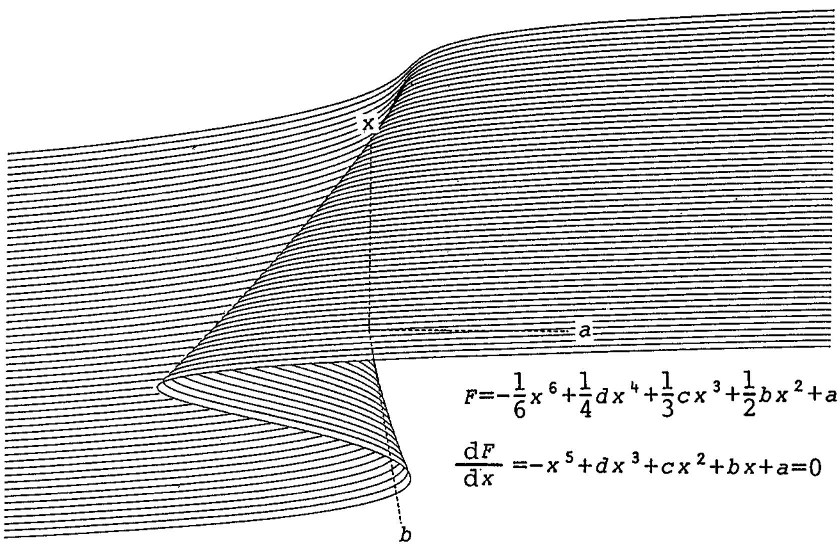


Fig. 8 Flow chart for drawing the swallowtail bifurcation set (2)

2.3 Butterfly Catastrophe

Butterfly catastrophe manifold for various values of c and d are shown from Fig.9 to Fig. 13. These are drawn in a similar procedure to Fig. 2. Control parameter a, b, c and d are called normal factor, splitting factor, bias factor and butterfly factor, respectively. Fig. 14 is the bifurcation set for various values of c and d. In this catastrophe, the state variable is trimodal for some values of control factors. In Fig.9 and Fig. 10, the statevariable is bimodal inside the bifurcation set. On the other hand, in Fig. 11, Fig. 12 and Fig. 13, the state variable is trimodal inside the shadowed area of Fig. 14 (c),(d),(e). This is the main feature of butterfly catastrophe. In general, it is difficult to understand the picture of butterfly catastrophe manifold corresponding to Fig. 11, Fig. 12 and Fig. 13, and such an exact and plain figure is not shown so far. Fig. 9 and Fig.10 are qualitatively the same with Fig. 1. In short, the cusp catastrophe is contained in the butterfly catastrophe. The butterfly catastrophe is not applied so frequently as the cusp catastrophe. In applying this catastrophe, it is important to understand its geometry accurately.



$$F=-\frac{1}{6}x^6+\frac{1}{4}dx^4+\frac{1}{3}cx^3+\frac{1}{2}bx^2+ax$$

$$\frac{dF}{dx}=-x^5+dx^3+cx^2+bx+a=0$$

Fig. 9 Butterfly catastrophe manifold (c $>0, d<0$)

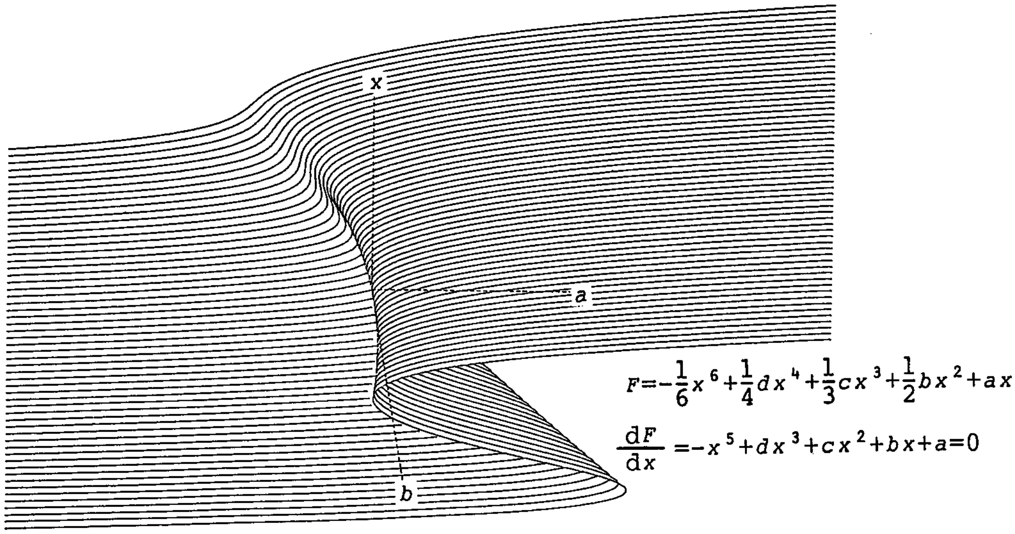


Fig. 10 Butterfly catastrophe manifold ($c<0, d<0$)

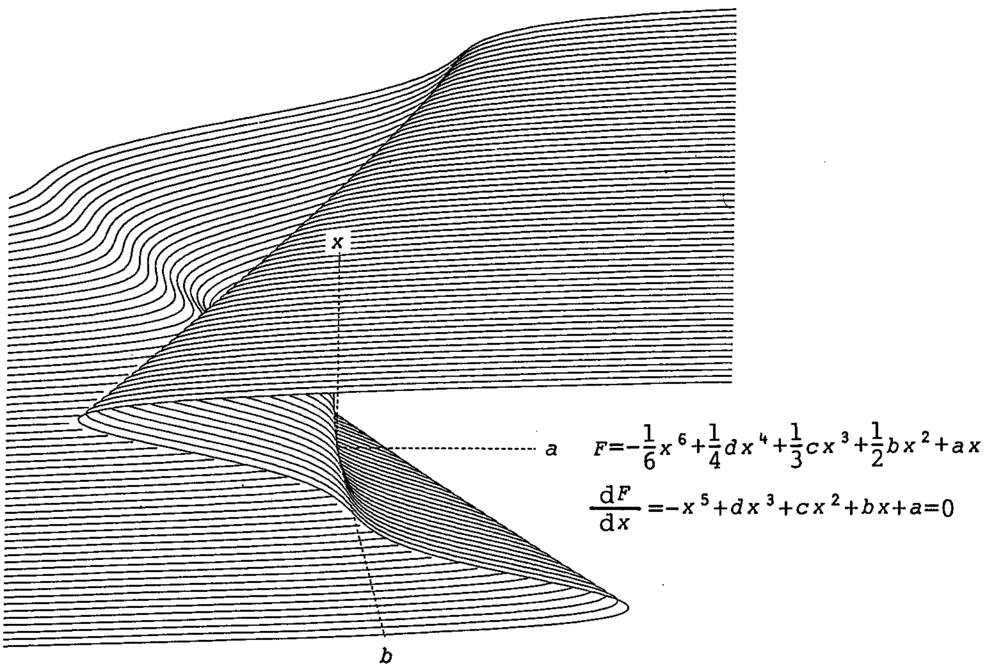


Fig. 11 Butterfly catastrophe manifold ($c = 0, d>0$)

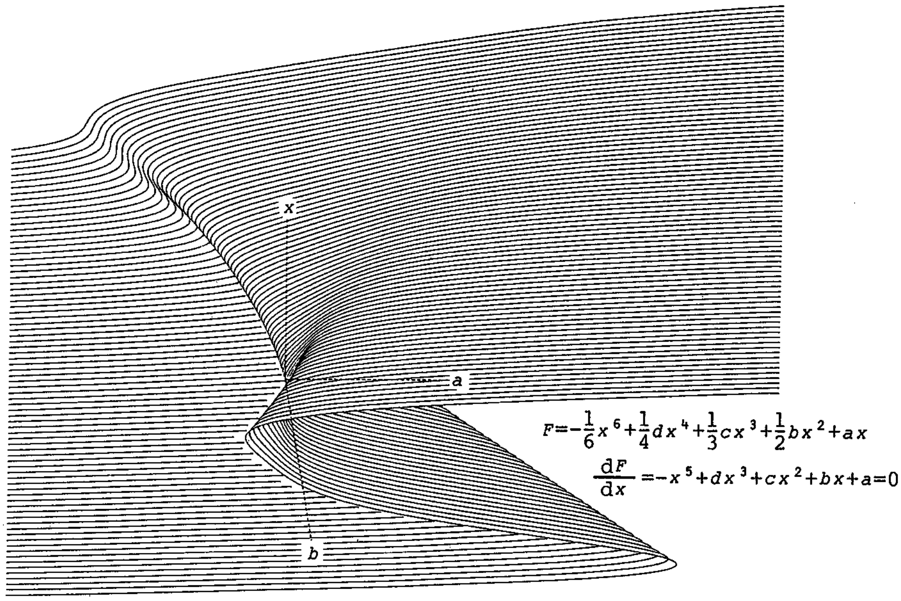


Fig. 12 Butterfly catastrophe manifold ($c>0, d>0$)

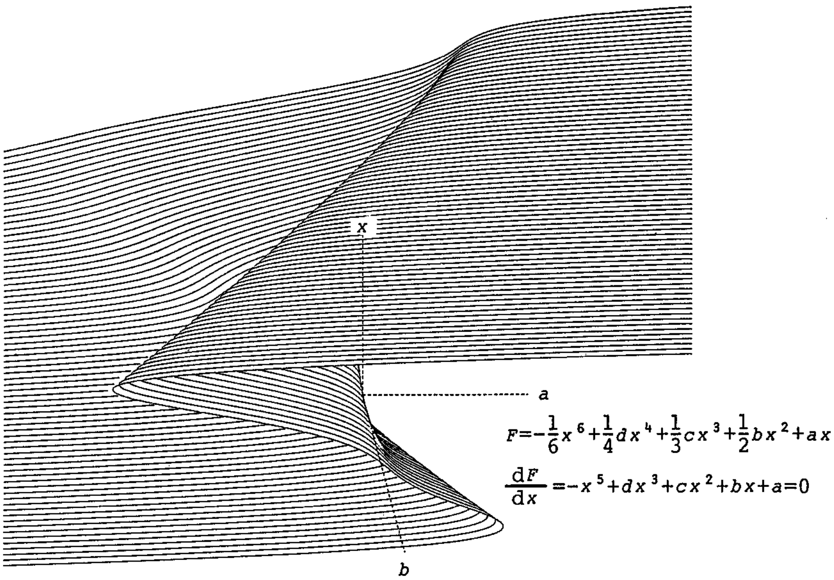
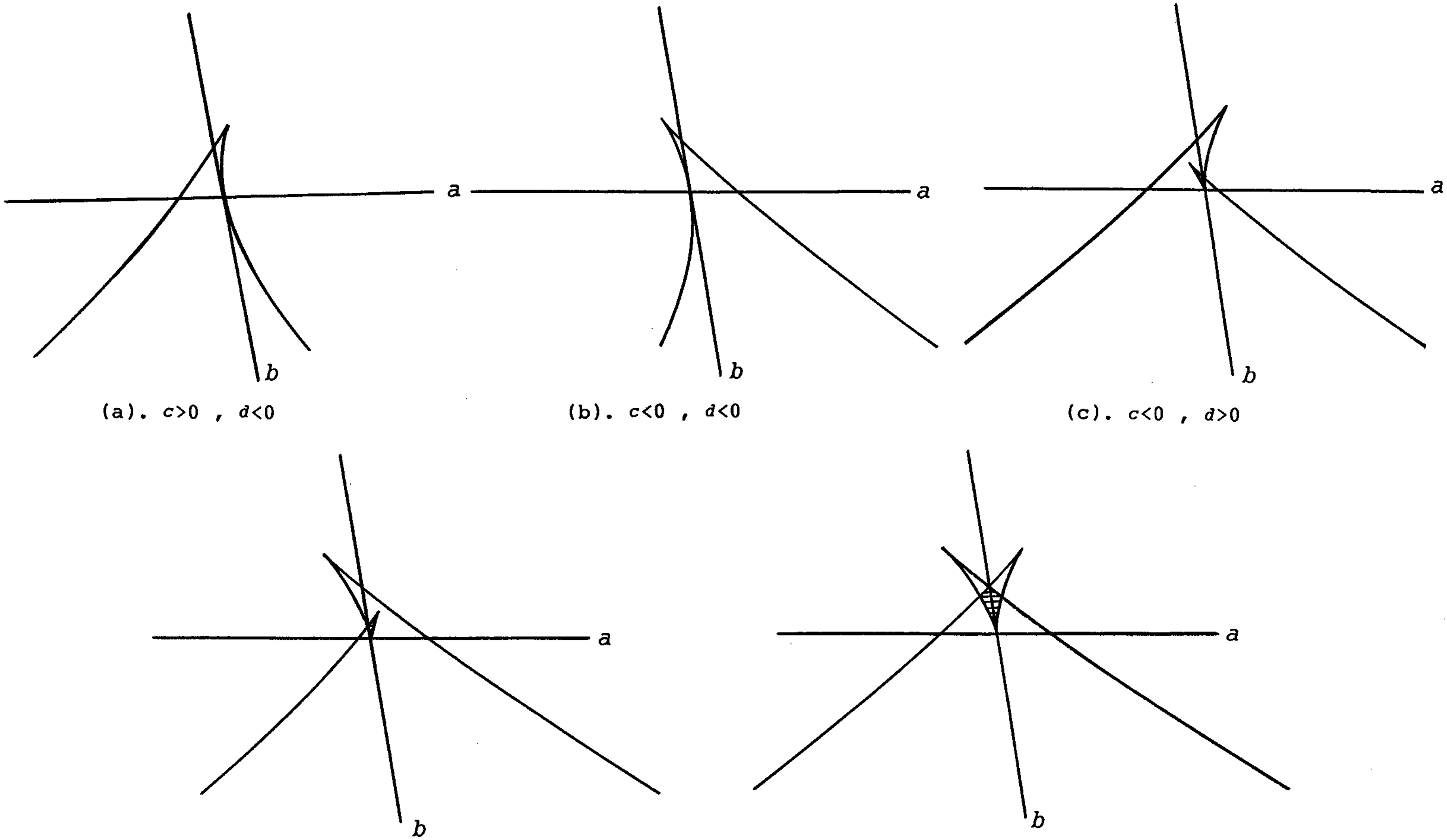


Fig. 13 Butterfly catastrophe manifold ($c<0, d>0$)



(a). $c>0$, $d<0$ (b). $c<0$, $d<0$ (c). $c<0$, $d>0$

(d). $c>0$, $d>0$ (e). $c=0$, $d>0$

Fig. 14 Butterfly bifurcation set

3. Conclusions

In this paper, the method for drawing the cuspoid catastrophe manifold and the swallowtail bifurcation set was shown, and their geometry was discussed. Aside from this, the method for representing the elementary catastrophe shown in Table. 1 was established. It is necessary to grasp the geometrical feature in the application of catastrophe model.

Table 1. Elementary catastrophe

number of State variable	codimension	Universal unfolding	name	representation method
1	1	$F=1/3x^2+ax$	fold	3-dimensional manifold
	2	$F=\pm 1/4x^4+1/2bx^2+ax$	cusp	3-dimensional manifold ruled surface 2-dimensional bifurcation set
	3	$F=1/5x^5+1/3cx^3+1/2bx^2+ax$	swallow-tail	3-dimensional manifold 3-dimensional bifurcation set ruled surface
	4	$F=\pm 1/6x^6+1/4dx^4+1/3cx^3+1/2bx^2+ax$	butterfly	3-dimensional manifold ruled surface 2-dimensional bifurcation set
2	3	$F=x^3+y^3+axy+bx+cy$	hyperbolic umbilic	2-dimensional bifurcation set ruled surface
	3	$F=x^3-xy^2+a(x^2+y^2)+bx+cy$	elliptic umbilic	2-dimensional bifurcation set ruled surface
	4	$F=\pm x^2y+y^4+ax^2+by^2+cx+dy$	parabolic umbilic	2-dimensional bifurcation set ruled surface

References

1) E.C. Zeeman, "Catastrophe Theory", 1st ed. p. 1, Addison-Wesley Publishing Company, Inc., Massachusetts (1977).
2) A. E. R. Woodcock and T. Poston, "A Geometrical Study of the Elementary Catastrophe'', 1st ed. p.133, Springer-Verlag, New York (1974).