Boundary Element Analysis of Bending Problems of Plates with Free or Fixed Edges

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(Received Nov. 15, 1983)

The bending and twisting moments and the transverse shearing force in a plate are derived from the "moment functions" U and V. The field of (U, V) becomes analogous to the displacement field of two-dimensional elasto-statics. The boundary conditions are such that, on an unsupported edge, the boundary values of U and V are known and that the derivatives of U and V are specified on a fixed edge. This is the most advantage of the method to the boundary element method. The moment components are obtained from the strain components of the corresponding plane elasticity.

Numerical calculation was carried out by boundary element method with linear elements. The results show that the BEM analysis can be applied precisely to the bending of the plate with free edges.

1. Introduction

Many workers have been trying to apply the Boundary Element Method (BEM) to the plate bending problems. Jaswon and Maiti[1] have applied their BEM formulation for a stress function in two-dimensional elasticity to bending problems of the uniformly loaded clamped or simply-supported plates. Maiti and Chakrabarty[2] and Hansen[3] have proposed methods suitable for the restricted classes of plate bending problems. In the plate bending problem, it is difficult to deal with the free-edge conditions because of the second- and third-order derivatives of the deflection function. The numerical difficulties also arise for the free-edge boundary conditions in the indirect BEM formulation.

More recently, Wu and Altiero[4] have suggested a little more general treatment by embedding the plate of interest in the circular clamped plate for which Green's function for an unit load is known. They have determined a fictitious line load and a normal moment along a contour which lies outside of the plate contour so as to satisfy the prescribed boundary conditions on the original boundary and avoided the second order singularities in the integrant of the equations. Stern[5] has employed a more direct approach and obtained a pair of integral equations involving displacement, normal slope, bending moment and equivalent shear on the boundary.

In the present work, an alternative approach is employed to obtain a formulation for a bending problem of a plate with free edges, which is apropriate to apply the BEM. The method is in spirit the same as the ones proposed by Southwell[6] and extended to plates of variable thickness and mixed boundary conditions by Fung[7] and further to the multiply-connected plates by Matsumoto and Sekiya[8]. The "moment functions", which was termed "stress functions" by Fung, are intro-

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duced in relation to the bending and twisting moments and the transverse shear in the plate. Then an analogy between the plate bending problem and the plate stretching problem is established much clearer than the former works. It is needed to change partially the BEM program of the two-dimensional elasticity to calculate the bending moments. Moments on the boundary are able to be computed from boundary data after some algebraic operations. Numerical examples show that the present method is very useful to analyze the moment distributions.

2. Analogy between bending- and stretching- problems of plates

2.1 Fundamental equations

The plate is assumed to be homogeneous and isotropic and to obey the Kirchhoff-Love assumption. Moreover, it is subjected to a distributed surface load q and/or an edge load Q_n or moments M_n or M_{nt} (Fig. 1). The equilibrium equations of this plate are well-known as

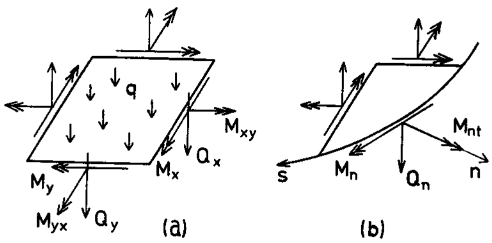

Fig. 1 Sign convention of moments and shearing forces.

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}-Q_x=0, \quad \frac{\partial M_{xy}}{\partial x}-\frac{\partial M_y}{\partial y}+Q_y=0$$
$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0 \tag{1}$$

We define "load functions", Ω_1 and Ω_2, such as

$$\frac{\partial^2 \Omega_1}{\partial x^2}+\frac{\partial^2 \Omega_2}{\partial y^2}=q \tag{2}$$

Then, equations (1) are identically satisfied by introducing "moment functions", U and V such that

$$Q_x=\frac{\partial \psi}{\partial y}-\frac{\partial \Omega_1}{\partial x}, \quad Q_y=-\frac{\partial \psi}{\partial x}-\frac{\partial \Omega_2}{\partial y} \tag{3}$$

$$M_x=\frac{\partial V}{\partial y}-\Omega_1, \quad M_y=\frac{\partial U}{\partial x}-\Omega_2 \tag{4}$$

and

$$M_{xy}=\frac{\partial U}{\partial y}+\psi, \quad M_{yx}=-\frac{\partial V}{\partial x}+\psi \tag{5}$$

Here, equations (5) can be rewritten as

$$M_{xy}=-M_{yx}=\frac{1}{2}\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right) \tag{6}$$

The equations governing U and V are the compatibility equations and can be derived from the relations between moments and deflection w, i.e.

$$\begin{aligned} &M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right), \quad M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right) \\ &M_{xy}=D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{7}$$

where D is the flexural rigidity of the plate and ν is Poisson's ratio. From equations (4), (6) and (7), we have

$$\begin{aligned} &\frac{\partial^2 w}{\partial y^2}=\frac{-1/D}{1-\nu^2}\left\{\frac{\partial U}{\partial x}-\nu\frac{\partial V}{\partial y}-(\Omega_2-\nu\Omega_1)\right\} \\ &\frac{\partial^2 w}{\partial x^2}=\frac{-1/D}{1-\nu^2}\left\{\frac{\partial V}{\partial y}-\nu\frac{\partial U}{\partial x}-(\Omega_1-\nu\Omega_2)\right\} \\ &-\frac{\partial^2 w}{\partial x \partial y}=\frac{-1/D}{2(1-\nu)}\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right) \end{aligned} \tag{8}$$

By eliminating w from above equations, the governing equations of new functions U and V become

$$\begin{aligned} &\frac{\partial}{\partial x}\left[\frac{-1/D}{1-\nu^2}\left(\frac{\partial U}{\partial x}-\nu\frac{\partial V}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\frac{-1/D}{2(1-\nu)}\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right)\right]-\frac{\partial}{\partial x}\left[\frac{-1/D}{1-\nu^2}(\Omega_2-\nu\Omega_1)\right]=0 \\ &\frac{\partial}{\partial y}\left[\frac{-1/D}{1-\nu^2}\left(\frac{\partial V}{\partial y}-\nu\frac{\partial U}{\partial x}\right)\right]+\frac{\partial}{\partial x}\left[\frac{-1/D}{2(1-\nu)}\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right)\right]-\frac{\partial}{\partial y}\left[\frac{-1/D}{1-\nu^2}(\Omega_1-\nu\Omega_2)\right]=0 \end{aligned} \tag{9}$$

Comparing equations (9) with the governing equations of the stretching problems of thin plates under the initial strains (see Appendix), it is concluded that both systems of equations become identical if E^* and ν^* are put in place of $-1/D$ and $-\nu$ respectively, and Ω_2 and Ω_1 are looked as the initial strains ε_{x0} and ε_{y0} respectively. Therefore we can interpret U and V as the components of a displacement and then the equivalent body force components X^* and Y^* are given by

$$X^*=\frac{1/D}{1-\nu^2}\frac{\partial}{\partial x}(\Omega_2-\nu\Omega_1), \quad Y^*=\frac{1/D}{1-\nu^2}\frac{\partial}{\partial y}(\Omega_1-\nu\Omega_2) \tag{10}$$

So, if the plane stress problem for the plate with body forces given by equations (10) can be solved, the strain distributions give the moment distributions of the original bending problems (see Eqs. (4) and (6)).

2.2 Boundary conditions

Values of U and V at the point on the boundary can be calculated as follows. The differentials of U and V on the boundary are given by

$$\mathrm{d}U=\left(-\frac{\partial U}{\partial x}m+\frac{\partial U}{\partial y}l\right)ds\,,\quad \mathrm{d}V=\left(-\frac{\partial V}{\partial x}m+\frac{\partial V}{\partial y}l\right)ds \tag{11}$$

where l and m are direction cosines of the outer normal $\boldsymbol{n}$ and s is an arc length along the boundary (Fig. 1). Substituting equations (4) and (5) into equations (11), we obtain

$$\begin{aligned} dU&=-(mM_y-lM_{xy}+l\psi-m\Omega_2)ds \\ dV&=(mM_{yx}+lM_x-m\psi+l\Omega_1)ds \end{aligned} \tag{12}$$

Using the following relations

$$\begin{aligned} mM_y-lM_{xy}&=mM_n-lM_{nt} \\ mM_{yx}+lM_x&=lM_n+mM_{nt} \end{aligned} \tag{13}$$

and integrating the results along the boundary, we obtain the boundary values of U and V as follows:

$$\begin{aligned} U&=U_0-\int_0^s(mM_n-lM_{nt}+l\psi+m\Omega_2)ds \\ V&=V_0+\int_0^s(lM_n+mM_{nt}-m\psi+l\Omega_1)ds \end{aligned} \tag{14}$$

Similarly, ψ is given from equations (3) as

$$\psi=\psi_0+\int_0^s\left(Q_n+l\frac{\partial\Omega_1}{\partial x}+m\frac{\partial\Omega_2}{\partial y}\right)ds \tag{15}$$

U_0, V_0 and ψ_0 are integration constants. This set of boundary conditions corresponds to the unsupported edge of the plate on which edge moments and shearing force are specified. Especially, $M_n=M_{nt}=Q_n=0$ along a free edge, and then the boundary values of U and V are given by

$$\begin{aligned} \psi&=\psi_0+\int_0^s\left(l\frac{\partial\Omega_1}{\partial x}+m\frac{\partial\Omega_2}{\partial y}\right)ds \\ U&=U_0-\int_0^s(l\psi+m\Omega_2)ds \\ V&=V_0-\int_0^s(m\psi-l\Omega_1)ds \end{aligned} \tag{16}$$

for a free edge.

Another important boundary condition of the plate bending problem is the fixed-edge condition, i.e. $w=\partial w/\partial n=0$. This implies $\partial w/\partial x=\partial w/\partial y=0$ and then

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial w}{\partial x}\right)=-m\frac{\partial^2 w}{\partial x^2}+l\frac{\partial^2 w}{\partial x\partial y}=0\,,\quad \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial w}{\partial y}\right)=-m\frac{\partial^2 w}{\partial x\partial y}+l\frac{\partial^2 w}{\partial y^2}=0 \tag{17}$$

Therefore, substituting equations (8) into above equations, the following boundary conditions are obtained:

$$p_x=\frac{-l/D}{1-\nu^2}(\Omega_2-\nu\Omega_1)$$
$$p_y=\frac{-m/D}{1-\nu^2}(\Omega_1-\nu\Omega_2) \tag{18}$$

where the following symbols are used:

$$p_x=\frac{-1/D}{1-\nu^2}\left[l\left(\frac{\partial U}{\partial x}-\nu\frac{\partial V}{\partial y}\right)+\frac{1+\nu}{2}m\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right)\right]$$
$$p_y=\frac{-1/D}{1-\nu^2}\left[m\left(\frac{\partial V}{\partial y}-\nu\frac{\partial U}{\partial x}\right)+\frac{1+\nu}{2}l\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right)\right] \tag{19}$$

These conditions are interpreted as the stress boundary conditions of the plane stress problem.

2.3 Extra conditions

Equations (14) or (16) can be rewritten as follows:

$$U=-C_1y+C_2+\overline{U},\quad V=C_1x+C_3+\overline{V} \tag{20}$$

Here, terms $\overline{U}$ and $\overline{V}$ are able to be derived from edge moments, shearing force and load functions Ω_1 and Ω_2. Constants C_1, C_2 and C_3 are different on every boundary contour of multiply-connected plate, but only one triplet may be set equal zero since the derivatives of U and V are needed to derive the moments (see equations (4) and (6)). These triplets of constants are to be determined for each contour Γ_i which contains the free edge. The conditions to determine these constants are the uniqueness conditions of the plate deflection w and its derivatives:

$$\int_{\Gamma_i}dw=0,\quad \int_{\Gamma_i}d\left(\frac{\partial w}{\partial x}\right)=0,\quad \int_{\Gamma_i}d\left(\frac{\partial w}{\partial y}\right)=0 \tag{21}$$

Using the following relations

$$d\left(\frac{\partial w}{\partial x}\right)=\frac{\partial^2 w}{\partial x^2}dx+\frac{\partial^2 w}{\partial x\partial y}dy,\quad d\left(\frac{\partial w}{\partial y}\right)=\frac{\partial^2 w}{\partial x\partial y}dx+\frac{\partial^2 w}{\partial y^2}dy$$
$$dw=d\left(x\frac{\partial w}{\partial x}\right)+d\left(y\frac{\partial w}{\partial y}\right)-xd\left(\frac{\partial w}{\partial x}\right)-yd\left(\frac{\partial w}{\partial y}\right) \tag{22}$$

and equations (8), equations (21) may be rewritten as

$$\int_{\Gamma_i}p_x ds=\frac{-1/D}{1-\nu^2}\int_{\Gamma_i}l(\Omega_2-\nu\Omega_1)ds$$
$$\int_{\Gamma_i}p_y ds=\frac{-1/D}{1-\nu^2}\int_{\Gamma_i}m(\Omega_1-\nu\Omega_2)ds \tag{23}$$
$$\int_{\Gamma_i}(xp_y-yp_x)ds=\frac{-1/D}{1-\nu^2}\int_{\Gamma_i}[m\cdot x(\Omega_1-\nu\Omega_2)-ly(\Omega_2-\nu\Omega_1)]ds$$

where p_x and p_y are quantities defined by equations (19). In the corresponding plane stress problem, these are the equilibrium conditions of the resultant force and moment in each hole.

2.4 Removal of multivaluedness of U and V

Sometimes it may happen that the boundary values of U and V derived from equation (16) are multi-valued. In this case, single-valued functions $\tilde{U}$ and $\tilde{V}$ are introduced as follows:

$$\begin{aligned}\tilde{U}&=U-\frac{1}{2\pi}\sum U_i \tan^{-1}\frac{y-y_i}{x-x_i}\\ \tilde{V}&=V-\frac{1}{2\pi}\sum V_i \tan^{-1}\frac{y-y_i}{x-x_i}\end{aligned} \tag{24}$$

where U_i and V_i are dislocations of U and V at point (x, y) on a contour Γ_i where U and V are specified (Fig. 2). The governing equations and the boundary values for the field $(\tilde{U}, \tilde{V})$ are obtained from equations (9) and (20). As a result, we have additional terms in equations (10), (18), (20) and (23).

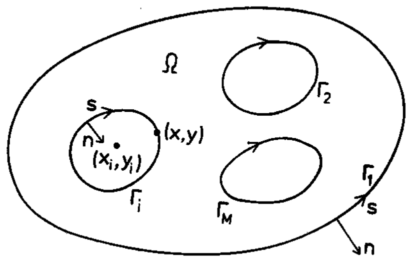

Fig. 2 Configuration of the plate with holes.

3. Application of the boundary element method

As shown above, the governing equations of moment functions are analogous to those of the plane stress problem. The free edges in the bending problems correspond to the edges on which displacements are specified, and the fixed edges to the stressed edges in the plane stress problem. But there are some differences between two problems, i.e. the unknown constants in equations (20) and the extra equations (23) to determine these constants. Therefore, if we intend to use the ordinary BEM program for plane elasto-static problems, it has to be modified a little.

Fundamental integral equations[9] for a two-dimensional elasto-statics are, using summation convention,

$$c_{ij}u_j=\int_\Gamma (u^*_{ij}p_j-p^*_{ij}u_j)\mathrm{d}\Gamma+\int_\Omega u^*_{ij}X_j\mathrm{d}\Omega \tag{25}$$

where Ω is the domain of the plate, $\Gamma=\Gamma_1+\Gamma_2+\cdots+\Gamma_M$ is the boundary of the plate, and u_j, p_j and X_j are the components of displacements, tractions and body forces, respectively. c_{ij} is the constant estimated by means of the rigid body conditions. u^*_{ij} is the fundamental solution which gives the displacement field of the in finite elastic medium subjected to an unit load at point ξ in x_j-direction. p^*_{ij} is the i-th traction at point x due to the field u^*_{ij}. In an usual sequence, discretizing equation (25) and using the boundary conditions, the resulted systems of equations are solved. But in our case, we have the specified values of displacement given by equations (20) and the extra conditions (23), then unknowns increase by three per such a contour. Equations (23) can be discretized with equation (25) in the same way as the ordinary BEM formulation, and they are solved with equations (18) and (20).

The derivatives at inner point x which are used to compute the moments (see equations (4) and (6)) are calculated from the equation

$$\frac{\partial u_i}{\partial x_j}=\int_\Gamma (f^*_{ijk}p_k-g^*_{ijk}u_k)\mathrm{d}\Gamma+\int_\Omega f^*_{ijk}X_k\mathrm{d}\Omega \tag{26}$$

where kernel functions f^* and g^* are given as

$$\begin{aligned} f^*_{ijk}&=\frac{1}{8\pi G(1-\nu_e)r}\left\{(3-4\nu_e)\frac{\partial r}{\partial x_j}\delta_{ik}-\frac{\partial r}{\partial x_k}\delta_{ji}-\frac{\partial r}{\partial x_i}\delta_{jk}+2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\right\}\\ g^*_{ijk}&=\frac{-1}{4\pi(1-\nu_e)r^2}\left\{2\frac{\partial r}{\partial n}\left[(1-2\nu_e)\frac{\partial r}{\partial x_j}\delta_{ik}-\frac{\partial r}{\partial x_k}\delta_{ij}-\frac{\partial r}{\partial x_i}\delta_{jk}+4\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\right]\right.\\ &\quad-\left[(1-2\nu_e)\delta_{ik}+2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_k}\right]n_j+(1-2\nu_e)\left(\delta_{ij}-2\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\right)n_k\\ &\quad\left.-(1-2\nu_e)\left(\delta_{jk}-2\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial x_k}\right)n_i\right\} \end{aligned} \tag{27}$$

and $G=-1/2D(1-\nu)$, $\nu_e=-\nu/(1-\nu)$, r is the distance from an inner point x to a point ξ on the boundary, n_k is the outer unit normal vector and δ_{ij} is Kronecker's delta. We can not calculate the derivatives at points on the boundary because of the strong singularities contained in the kernels. However the strains at the boundary points can be calculated from the boundary data as follows (Fig. 3).

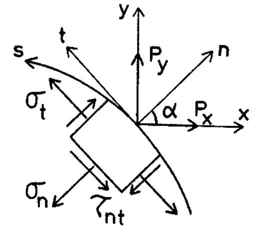

Fig. 3 Stresses on the boundary.

The normal and shear stresses and the tangential strain are computed from the boundary data as

$$\begin{aligned} \sigma_n&=p_x\cos\alpha+p_y\sin\alpha\\ \tau_{nt}&=-p_x\sin\alpha+p_y\cos\alpha\\ \varepsilon_t&=-\frac{du}{ds}\sin\alpha+\frac{dv}{ds}\cos\alpha \end{aligned} \tag{28}$$

Then the rest of the strain components are obtained by means of the stress-strain relations as follows:

$$\begin{aligned} &\sigma_t=(2G\varepsilon_t+\nu_e\sigma_n)/(1-\nu_e) \\ &\varepsilon_n=((1-2\nu_e)\sigma_n/2G-\nu_e\varepsilon_t)/(1-\nu_e) \\ &\gamma_{nt}=\tau_{nt}/G \end{aligned} \tag{29}$$

4. Numerical examples

Numerical analyses were carried out by means of BEM with linear elements. First example is shown in Fig. 4. This is one of the most difficult problem to analyze because the plate has the free edges and moreover is subjected to the concentrated load P. We put $\Omega_1=\Omega_2=0$, since $q=0$ and assume $\psi_0=-P/2$, $U_0=0$, $V_0=aP/2$ in equations (16) to get the symmetric boundary conditions as

$$\begin{aligned} &p_x=p_y=0 && \text{on fixed edge } x=0 \\ &U=0\,,\quad V=\frac{P}{2}(x-a)\,\mathrm{sgn}\,(y) && \text{on free edge } y=\pm b/2 \\ &U=\frac{P}{2}\left(\frac{b}{2}-|y|\right),\quad V=0 && \text{on free edge } x=a \end{aligned} \tag{30}$$

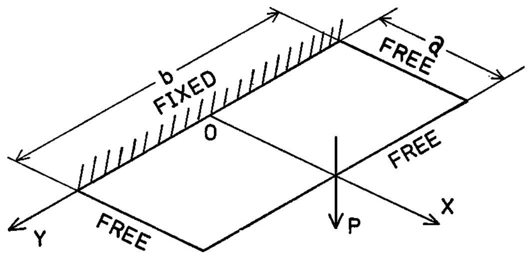

Fig. 4 Rectangular plate with three edges free and a fourth edge fixed.

The bending moment distribution on the fixed edge $x=0$ is shown in Fig. 5 with the results of the photo-elastic experiment[10]. The number N of the boundary element is 20 or 30.

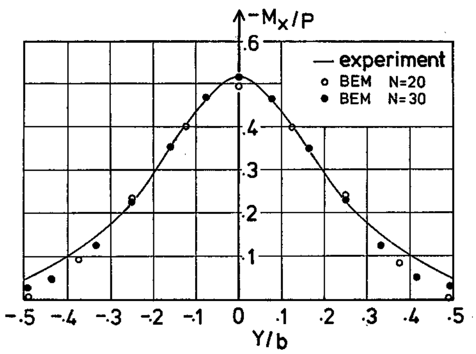

Fig. 5 Bending moment distributions on the fixed edge ($b/a=4$).

The second example is the bending of the square plate with three edges fixed and a fourth edge free (Fig. 6–a). Two loading cases, a uniformly distributed load (case I) and a hydrostatically distributed load (case II), were considered (Fig. 6–b and c). Since the surface load q is applied to the plate, we need to find out

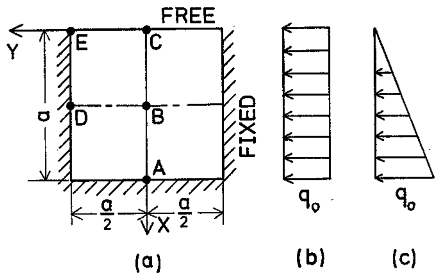

Fig. 6 Geometry of the plate and the loading form.

the load functions Ω_i in equation (2). We assumed $\Omega_1=q_0x^2/4$ and $\Omega_2=q_0y^2/4$ for case I. Then the equivalent body force is obtained from equation (10) as

$$X^*=-\nu q_0x/2D(1-\nu^2)\,,\quad Y^*=-\nu q_0y/2D(1-\nu^2) \tag{31}$$

and the surface traction applied to the fixed edge is given as

$$p_x=-lq_0(y^2-\nu x^2)/4D(1-\nu^2)\,,\quad p_y=-mq_0(x^2-\nu y^2)/4D(1-\nu^2) \tag{32}$$

and the prescribed displacement on the free edge is assumed as

$$U=V=0 \tag{33}$$

On the other hand, we assumed $\Omega_1=q_0x^3/6a$, $\Omega_2=0$ for case II. Thus the body force becomes

$$X^*=-\nu q_0x^2/2aD(1-\nu^2)\,,\quad Y^*=0 \tag{34}$$

and the boundary conditions are given as

$$U=V=0 \tag{35}$$

on a free edge and

$$p_x=\nu lq_0x^3/6aD(1-\nu^2)\,,\quad p_y=mq_0x^3/6aD(1-\nu^2) \tag{36}$$

on a fixed edge. The obtained results for the typical points shown in Fig. 6–a are compared in Table 1 with the values from Ref. 11.

To assure applicability of the present method to the perforated plate, the numerical analysis was carried out for the circular ring plate with fixed outer edge. The bending analysis also become difficult when the plate has holes, but fortunately the exact solutions are easily obtained in this case. When the moment M_0 is uniformly distributed along the edge of the central hole, the boundary condition is given as

$$\begin{aligned} &U=-C_1y+C_2-M_0x\,,\quad V=C_1x+C_3-M_0y && \text{on the inner edge} \\ &p_x=p_y=0 && \text{on the outer edge} \end{aligned} \tag{37}$$

Table 1 Moments of square plates with three edges fixed and a fourth edge free. $\bar{M}=M/q_0a^2$, $\nu=1/6$

Load case	N	A	B		C	D	E
		$\bar{M}_x$	$\bar{M}_x$	$\bar{M}_y$	$\bar{M}_y$	$\bar{M}_y$	$\bar{M}_y$
I	24	−0.0588	0.0151	0.0308	0.0460	−0.0613	−0.0889
	40	−0.0584	0.0140	0.0304	0.0440	−0.0641	−0.0880
	64	−0.0555	0.0135	0.0307	0.0434	−0.0660	−0.0859
	finite difference[11)]	−0.0510	0.0138	0.0317	0.0444	−0.0614	−0.0853
II	24	−0.0376	0.0109	0.0122	0.0106	−0.0262	−0.0153
	40	−0.0367	0.0097	0.0126	0.0099	−0.0283	−0.0133
	64	−0.0360	0.0092	0.0127	0.0096	−0.0291	−0.0117
	finite difference[11)]	−0.0299	0.0094	0.0135	0.0095	−0.0269	−0.0146

and the extra conditions to determine the constants C_i are

$$\oint p_x \mathrm{d}s=0\,, \quad \oint p_y \mathrm{d}s=0\,, \quad \oint (xp_y-yp_x)\mathrm{d}s=0 \tag{38}$$

for the inner boundary. The numerical calculation was carried out to the case which the ratio of the outer and the inner radii (b/a) is equal to 2 and the number of the boundary element is 24 per each contour. The obtained moments are compared with the exact one (Fig. 7).

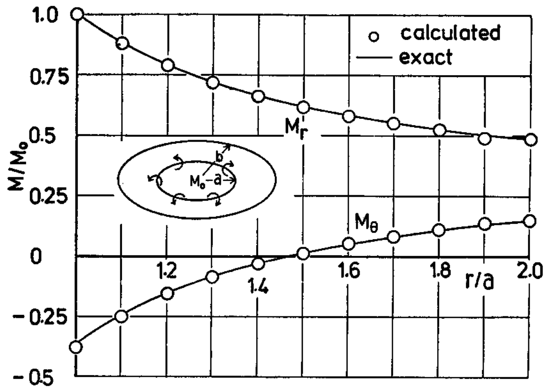

Fig. 7 Bending moment distributions on the radial line in the concentric circular plate under the uniform bending moment M_0 at the hole ($b/a=2$, 48 elements).

Next we consider the case that the plate is subjected to the uniform surface load q and the hole is free from stress. As load functions, we take $\Omega_1=qx^2/4$ and $\Omega_2=qy^2/4$. But in this case, the displacements on the inner boundary have discrepancies $U_1=-qa^2\pi y$ and $V_1=qa^2\pi x$ where $x^2+y^2=a^2$. Then we introduce a single-valued displacement such as

$$\tilde{U}=U+qa^2y\frac{1}{2}\tan^{-1}\frac{y}{x}\,, \quad \tilde{V}=V-qa^2x\frac{1}{2}\tan^{-1}\frac{y}{x} \tag{39}$$

For this system, we have the body force;

$$\tilde{X}=\frac{qx}{2D(1-\nu^2)}\left\{\frac{(1+\nu)a^2}{x^2+y^2}-\nu\right\}$$
$$\tilde{Y}=\frac{qy}{2D(1-\nu^2)}\left\{\frac{(1+\nu)a^2}{x^2+y^2}-\nu\right\} \tag{40}$$

the boundary conditions;

$$\tilde{U}=-C_1y+C_2-q(x^3+3a^2x-4a^3)/12$$
$$\tilde{V}=C_1x+C_3-q(y^3+3a^2y)/12 \qquad x^2+y^2=a^2 \tag{41}$$

along the edge of the central hole and

$$\tilde{p}_x=\frac{-qx}{4D(1-\nu^2)b}(y^2-\nu x^2+2\nu a^2)$$
$$\tilde{p}_y=\frac{-qy}{4D(1-\nu^2)b}(x^2-\nu y^2+2\nu a^2) \qquad x^2+y^2=b^2 \tag{42}$$

along the outer contour and the extra conditions;

$$\oint \tilde{p}_x \mathrm{d}s=0, \quad \oint \tilde{p}_y \mathrm{d}s=0, \quad \oint (x\tilde{p}_y-y\tilde{p}_x)\mathrm{d}s=0 \tag{43}$$

for the inner boundary. The plate ($b/a=2$) was divided into 200 triangular small cells and 48 boundary elements for a numerical computation. Fig. 8 shows the bending moment distributions obtained by BEM and the theoretical research.

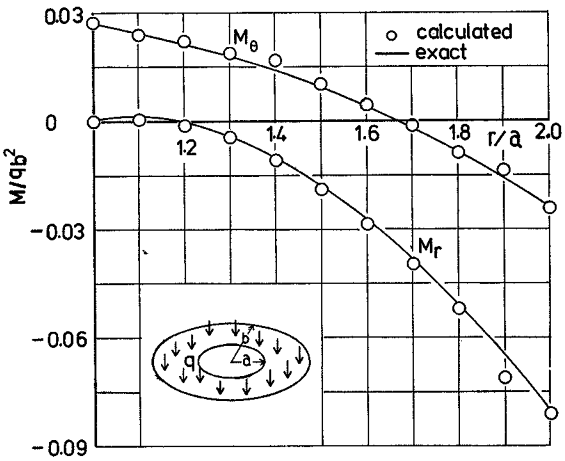

Fig. 8 Bending moment distributions on the radial line in the concentric circular plate with a free circular hole under the uniformly distributed load q ($b/a=2$, 48 elements and 200 internal cells).

The last example is the bending of the clamped square plate with a free central circular hole under the uniformly distributed lateral load q. We have the same systems as the former one except the conditions (42). The tractions along the outer contour is given as

$$\begin{aligned}\tilde{p}_x&=\frac{-q}{4D(1-\nu^2)}\left[l(y^2-\nu x^2)-\frac{2a^2}{x^2+y^2}\{l(y^2-\nu x^2)-(1+\nu)mxy\}\right]\\ \tilde{p}_y&=\frac{-q}{4D(1-\nu^2)}\left[m(x^2-\nu y^2)-\frac{2a^2}{x^2+y^2}\{m(x^2-\nu y^2)-(1+\nu)lxy\}\right]\end{aligned} \tag{44}$$

Since no exact solution is available, the bending test was carried out to the steel plate whose side length L is 500 mm, and the radius of the central hole a is 100 mm. The bending moments on the x-axis obtained by BEM are compared with the experimental results in Fig. 9.

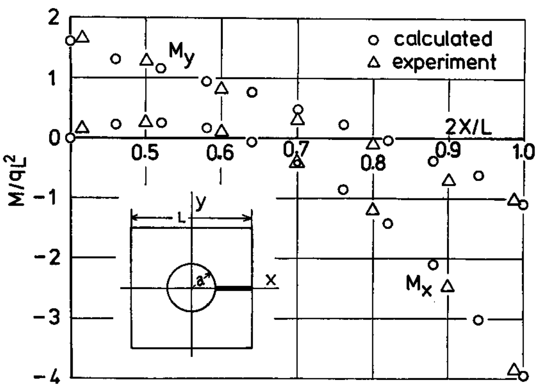

Fig. 9 Bending moment distributions on the x-axis in the square plate with a free circular hole under the uniformly distributed load q ($a/L=0.2$, 48 elements and 200 internal cells).

5. Concluding remarks

In the analysis of the bending of the plate, the existence of the free edges or the holes makes it difficult to find out the deflection function satisfying the boundary conditions. The free edge is also the neck point in the analysis of the plate by BEM.

In this paper, the analogy between the plate bending problem and the plate stretching problem was made much clearer than the former work, and for the analysis of the perforated plate it was pointed out that the dislocations, when we go around the hole, are dependent on the point where we start. Furthermore, we brought this analogy into the boundary element method. As illustrated in the above chapter, the results are sufficiently enough for us.

We believe that we can make the best use of BEM in the field of the bending analysis of the perforated plate with free edges.

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Appendix

The equilibrium equations for two-dimensional elastic problem is

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+X=0\,, \quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+Y=0 \tag{A-1}$$

where x, y are cartesian coordinates, σ_x, σ_y, τ_{xy} are the components of stress and X, Y are the components of body force. The stress-strain relations of a homogeneous, isotropic medium with initial strains ε_{x0}, ε_{y0} in the plane stress state are given as

$$\begin{aligned}
\sigma_x&=\frac{E}{1-\nu^2}\{\varepsilon_x+\nu\varepsilon_y-(\varepsilon_{x0}+\nu\varepsilon_{y0})\}\\
\sigma_y&=\frac{E}{1-\nu^2}\{\varepsilon_y+\nu\varepsilon_x-(\varepsilon_{y0}+\nu\varepsilon_{x0})\}\\
\tau_{xy}&=\frac{E}{2(1+\nu)}\gamma_{xy}
\end{aligned} \tag{A-2}$$

where E is Yong's modulus and ν is poisson's ratio.

Substituting (A–2) into (A–1) and using the relations between strain and displacement (u, v)

$$\varepsilon_x=\frac{\partial u}{\partial x}\,, \quad \varepsilon_y=\frac{\partial v}{\partial y}\,, \quad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} \tag{A-3}$$

we obtain the governing equations with respect to displacement as

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\frac{E}{1-\nu^2}\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\frac{E}{2(1+\nu)}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+X-\frac{\partial}{\partial x}\left[\frac{E}{1-\nu^2}(\varepsilon_{x0}+\nu\varepsilon_{y0})\right]=0\\
&\frac{\partial}{\partial y}\left[\frac{E}{1-\nu^2}\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right)\right]+\frac{\partial}{\partial x}\left[\frac{E}{2(1+\nu)}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+Y-\frac{\partial}{\partial y}\left[\frac{E}{1-\nu^2}(\varepsilon_{y0}+\nu\varepsilon_{x0})\right]=0
\end{aligned} \tag{A-4}$$

The last term in the left-hand side is equivalent to the body force and then equivalent body force is

$$X^*=\frac{-E}{1-\nu^2}\frac{\partial}{\partial x}(\varepsilon_{x0}+\nu\varepsilon_{y0})\,, \quad Y^*=\frac{-E}{1-\nu^2}\frac{\partial}{\partial y}(\varepsilon_{y0}+\nu\varepsilon_{x0}) \tag{A-5}$$