Combinatorial Properties of Identifying Dominant Failure Paths in Structural Systems

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This paper clarifies the combinatorial properties of failure paths in structural systems. A branch-and-bound algorithm is proposed for selecting the stochastically dominant failure paths, and its characteristics are investigated. A modified algorithm is also provided which uses the lower and upper bounds of the failure path probabilities. Effects of the approximation are discussed on the resulting selected failure paths.

1. Introduction

There are many studies made of the reliability analysis of structural systems. At first, their concerns were devoted to the estimation of reliability bounds[1~6] for given modes of failure. Recently, it has been noticed[7~27] that identification of relevant failure modes for given structures is one of the keys to the development and application of structural reliability theory. A failure mode is often characterized as a set of failed elements (members, sections, *etc.*) which makes the structural system fail. Consequently, there exist many combinations of the failed members to yield structural failure, *i.e.*, failure modes.

A simplest way for selecting the relevant failure modes is the enumeration method which produces and evaluates all the possible modes. The method can not be applied to large structures with many degrees of redundancy since the number of the modes is astronomically large. This motivated the studies of strategies to find the dominant modes by searching only for a subset of all the failure modes. One of these approaches is to determine the most dominant failure modes among all the linear combinations of the predetermined independent failure mechanisms by using a nonlinear programming technique[18], whose applicability seems to be limited due to the state of arts of the optimization technique. An alternative is to select the probable failure paths[8~15], *i.e.*, sequences of failed elements to yield structural failure, by using a branch-and-bound concept[13~15),22,23,26,27] in combinatorial mathematics and then to integrate[22,26] them to relevant failure modes, when necessary.

The purpose of this paper is to clarify the combinatorial properties of the branch-and-bound approach for selecting the stochastically dominant failure paths in structural systems. At first, the properties of the failure paths are examined from a viewpoint of combinatorial mathematics. Then, a branch-and-bound algorithm is proposed and its characteristics are investigated. Finally, a modified branch-and-bound algorithm by using the lower and upper bounds of the failure path probabilities is proposed and its effects on the selected failure paths are discussed. Further, some concluding remarks are provided.

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2. Combinatorial Properties of Failure Paths

Consider a structural system which has n potential plastic hinges and s degrees of redundancy. Structural failure is defined as formation of a collapse mechanism in the system. Plastic hinges are assumed to develop one by one up to some specific number p_k until a collapse mechanism is formed. The sequence of those plastic hinges to form a collapse mechanism is symbolically denoted as $r_1, r_2, \cdots, r_p, \cdots,$ and r_{p_k}, which is called a complete failure path and the number of the plastic hinges p_k is a length of the failure path. On the other hand, the sequence of the plastic hinges which do not yield structural failure, *e.g.*, the failure path $r_1 \rightarrow r_2 \rightarrow \cdots \rightarrow r_p$ $(p<p_k)$ is called a partial failure path. Consider a case where a collapse mechanism is still formed even if some plastic hinges are removed from a complete failure path, for example r_p in the complete failure path $r_1 \rightarrow r_2 \rightarrow \cdots \rightarrow r_p \rightarrow \cdots \rightarrow r_{p_k}$. A plastic hinge such as r_p is called a redundant one while those which can not be removed from a complete failure path to form a collapse mechanism are called essential ones. The probability $P_{fp(q)}{}^{(p)}$ of a failure path $r_1 \rightarrow r_2 \rightarrow \cdots \rightarrow r_p$ is calculated as

$$P_{fp(q)}{}^{(p)} = P[\bigcap_{i=1}^{p} F_{r_i(q)}^{(i)}]$$

where $F_{r_i(q)}^{(i)}$ is the failure event that plastic hinge r_i develops at the i-th order of sequence. Superscript p denotes the length of the failure path and q is used to denote a particular failure path. When $p<p_k$, $P_{fp(q)}{}^{(p)}$ is the probability of a partial failure path while it is the probability of a complete failure path for $p=p_k$.

Theorem 1 (Essential plastic hinges)

The plastic hinge at the final failure stage of a complete failure path is an essential plastic hinge. The number r of the essential plastic hinges is given by

$$1 \leq r \leq s+1 \tag{1}$$

Proof

If the last developed plastic hinge is removed from the complete failure path, the failure path becomes a partial failure path from the definition of the complete failure path. The lower bound on the number of the essential plastic hinges corresponds to the collapse mechanism formed by development of one plastic hinge. The upper bound is the case when collapse occurs after losing all the degrees of redundancy.

Theorem 2 (Number of redundant plastic hinges)

For a complete failure path with length l, there exist at most $(l-1)$ redundant plastic hinges.

Proof

Consider the case where all the plastic hinges other than the last developed plastic hinge are not essential.

Theorem 3 (Length of complete paths)

The length $l(r)$ of the failure paths with r essential plastic hinges is bounded by

$$r \leq l(r) \leq s+1 \quad (1 \leq r \leq s+1) \tag{2}$$

Proof

The lower bound corresponds to the case where the failure path comprises solely the essential plastic hinges while the upper bound the case where the first $(s-r+1)$ plastic hinges are redundant.

Theorem 4 (Redundant plastic hinges)

Let there be r essential plastic hinges which allow $k(0 \leq k \leq s+1-r)$ redundant plastic hinges to form a complete failure path. Then, any subset of the k redundant plastic hinges constitutes a complete failure path which contains the r essential plastic hinges.

Proof

As seen from the definition of redundant plastic hinges, any number of the redundant plastic hinges contained in a complete failure path can be removed from the failure path. This means that it is always possible to form a complete failure path which contains the r essential plastic hinges and any subset of the k redundant plastic hinges.

Theorem 5 (Number of complete failure path (1))

Let there be r essential plastic hinges which allow a complete failure path with length l $(r \leq l \leq s+1)$, *i.e.*, there are $(l-r)$ redundant plastic hinges. The total number $N(r, l)$ of the complete failure paths comprising the r essential plastic hinges and any combination of the $(l-r)$ redundant plastic hinges is given by

$$N(r, l) = \sum_{k=0}^{l-r} r(k+r-1)!C(l-r, k) \tag{3}$$

$$= \sum_{k=0}^{l-r} r!(l-r)!C(k+r-1, k)/(l-r-k)! \tag{4}$$

Proof

Consider a case where complete failure paths contain k redundant plastic hinges $(k=0, 1, 2, \cdots, l-r)$. For the case, there exist $r(k+r-1)!C(l-r, k)$ different failure paths, which results by considering the number of the complete failure paths with r essential plastic hinges and k redundant ones $(r(k+r-1)!)$ and the number of combinations for selecting k redundant plastic hinges from $(l-r)$ hinges $(C(l-r, k))$. Then, summing up over all k yields Eq. (3). By simple algebraic manipulation, Eq. (4) results.

Theorem 6 (Number of complete failure paths (2))

$N(r, l)$ is an increasing function of l.

Proof

$$N(r, l+1) = \sum_{k=0}^{l+1-r} r(k+r-1)!C(l+1-r, k)$$

$$= \sum_{k=0}^{l-r} r(k+r-1)!C(l+1-r, k) + r(l+r)! \tag{5}$$

$$>\sum_{k=0}^{l-r} r(k+r-1)!C(l-r, k)=N(r, l) \tag{6}$$

The inequality (6) follows since

$$C(l+1-r, k)>C(l-r, k), \quad r(l+r)!>0 \tag{7}$$

Theorem 7 (Bound of the number of the complete failure paths (1))

$$N(r, s+1)\leq(s+1)! \tag{8}$$

Proof

By putting $l=s+1$ in Eq. (4) and considering the inequality $(s-r+1-k)!\geq 1$, the following relation follows

$$N(r, s+1)\leq r!(s-r+1)!(\sum_{k=0}^{s-r+1} C(k+r-1, k) \tag{9}$$

$$=r!(s-r+1)!C(s+1, s-r+1) \tag{10}$$

$$=(s+1)!$$

In deriving Eq. (10), the rule of sum[30] for the combinations is applied, *i.e.*,

$$C(n+1, m)=\sum_{k=0}^{m} C(n-k, m-k)=\sum_{k=0}^{m} C(k+n-m, k) \quad (m\leq n) \tag{11}$$

Theorem 8 (Bound of the number of the complete failure paths (2))

The total number of the complete failure paths is not larger than $P(n, s+1)=n(n-1)\cdots(n-s)$.

Proof

From **Theorems 6** and **7**, the total number of the complete failure paths with $r(\leq s)$ essential plastic hinges is evidently bounded from above by considering those of all the fictive complete failure paths which include the r essential plastic hinges and whose length is $(s+1)$. Consequently, the maximum number of the complete failure paths is attained when all the collapse mechanisms occur if and only if any combination of $(s+1)$ plastic hinges develop in the system.

Theorem 9 (Monotonicity of the probabilities of failure paths)

The probability of a partial failure path is a monotonically decreasing function of the failure stage, *i.e.*,

$$P_{fp(q)}{}^{(p)}\geq P_{fp(q)}{}^{(p+1)} \quad (p=1, 2, \cdots, p_k-1) \tag{12}$$

Proof

$$P_{fp(q)}{}^{(p+1)}=P[\bigcap_{i=1}^{p+1} F_{r_i(q)}^{(i)}]$$

$$\leq P[\bigcap_{i=1}^{p} F_{r_i(q)}^{(i)}]=P_{fp(q)}{}^{(p)} \tag{13}$$

3. A Branch-and-Bound Algorithm

A branch-and-bound concept[28] developed for finding the optimum combination in combinatorial problems is applied to select stochastically dominant failure paths. The following nomenclature is used in the description of the proposed branch-and-bound algorithm.

P_{fpm}=the maximum of the probabilities of the selected complete failure paths

$P_{fp(q)}^{(p_k)}$=the probability of a selected complete failure path

X=the set of the failure paths to be selected for branching

X_c=the set of the selected complete failure paths

X_t=the set of the discarded failure paths

x_c=a selected complete failure path

δ=a bounding constant

ϕ=a null set

x_0=the artificial starting point of the failure path which can proceed to any one of the potential plastic hinges

A branch-and-bound algorithm for selecting stochastically dominant failure paths

Step 1: —*initializing*—
Set $P_{fpm}=0$, $X_c=\phi$, $X_t=\phi$, and $X=\{x_0\}$. x_0 is specified as a path for partitioning.

Step 2: —*partitioning*—
1. Proceed one failure stage by adding each of all the potential plastic hinges to the specified partial failure path. The resulting failure paths are added to the set X of the failure paths to be selected for branching.
2. Evaluate the probabilities of the new failure paths.

Step 3: —*branching*—
1. Select the failure path with maximum probability among the newly partitioned failure paths.
2. Check the formation of a collapse mechanism.
3. If a collapse mechanism is formed, go to Step 4 for bounding by adding the selected failure path x_c to the set X_c of the selected complete failure paths. If not, go to *Step* 2 for further partitioning by specifying the selected failure path as the failure path to be partitioned.

Step 4: —*bounding*— (*discarding*)
1. Update the maximum P_{fpm} of the probabilities of the selected complete failure paths by setting $P_{fpm}=P_{fp(q)}^{(p_k)}$ when $P_{fpm}< P_{fp(q)}^{(p_k)}$
2. Discard the failure paths which have the failure probabilities smaller than $10^{-\delta}P_{fpm}$. Add the discarded failure paths to the set X_t of the discarded failure paths. Exclude the discarded failure paths and the selected complete failure path from the set X of the failure paths for branching. Consequently, the set X of the failure paths to be selected for branching is changed to

$$X \leftarrow X - X_t - X_c$$

Step 5:
If $X=\phi$, *i.e.*, there are no failure paths for branching, the

—completing— search is completed. If not, go to step 3–2 for further branching with the most probable failure path selected in the set X.

The proposed branch-and-bound procedure has the following properties.

Theorem 10 (Number of branching operations)

The number N_b of branching operations for the branch-and-bound method is bounded by

$$r_{min} \leq N_b \leq P(n, s+1)(s+1) \tag{14}$$

where $r_{min}(\leq s+1)$ is the minimum number of essential plastic hinges required to form a collapse mechanism.

Proof

The lower bound is attained when the minimum complete failure path corresponding to the complete failure path with the minimum length r_{min} is most probable and it is selected at the first. The upper bound corresponds to the case when all the complete failure paths have the length $(s+1)$ and the most probable path is selected in the last.

Theorem 11 (Selected failure paths (1))

All the failure paths which have the probabilities larger than / or equal to $10^{-\delta}P_{fpm}$ are selected by the branch-and-bound algorithm.

Proof

From the bounding criterion and the completion condition of the algorithm, there is no possibility that the failure paths with probabilities larger than $10^{-\delta}P_{fpm}$ are neglected.

Theorem 12 (Discarded failure paths (1))

A partial failure path which has the probability smaller than $10^{-\delta}P_{fpm}$ is excluded from the set of the failure paths to be selected for branching.

Proof

From the monotonicity of the failure path probability, the failure paths obtained from partitioning the partial failure paths do not have the probability larger than $10^{-\delta}P_{fpm}$. Consequently, they can be neglected.

Theorem 13 (The most probable failure path (1))

The complete failure path which gives the maximum of the probabilities of the complete failure paths, *i.e.*, P_{fpm}, at the time of the completion, is the most probable failure path.

Proof

The selection process by the branch-and-bound method is completed only when no failure paths with probabilities larger than $10^{-\delta}P_{fpm}$ $(\delta \geq 0)$ exist, and P_{fpm} is the maximum of the probabilities of the complete failure paths so far obtained.

Consequently, the statement of the theorem is evident.

4. Approximation of Failure Path Probabilities

Consider the case when the failure path probabilities $P_{fp(q)}{}^{(p)}$ are estimated by their lower and upper bounds, $P_{fp(q)(L)}{}^{(p)}$ and $P_{fp(q)(U)}{}^{(p)}$, *i.e.*,

$$P_{fp(q)(L)}{}^{(p)} \leq P_{fp(q)}{}^{(p)} = P[\bigcap_{i=1}^{p} F_{r_i(q)}^{(i)}] \leq P_{fp(q)(U)}{}^{(p)} \tag{15}$$

For example, these bounds are given by (see *Appendix*)

$$P_{fp(q)(U)}{}^{(p)} = \min_{j \in \{2,\cdots,p\}} P[F_{r_1(q)}^{(1)} \cap F_{r_j(q)}^{(j)}] \tag{16}$$

$$P_{fp(q)(L)}{}^{(p)} = \max\{0,\ P[F_{r_1(q)}^{(1)}] - P[F_{r_1(q)}^{(1)} \cap \bar{F}_{r_2(q)}^{(2)}] - \sum_{i=3}^{p} \min(P_{fp(q)(U)}^{(j-1)},\ P[F_{r_1(q)}^{(1)} \cap \bar{F}_{r_j(q)}^{(j)}])\} \tag{17}$$

Of course, more sophisticated bounds[4~6),29)] are also applicable. By using the bounds, the branch-and-bound algorithm is modified as follows.

The maximum of the lower bounds $P_{fp(q)(L)}{}^{(p_k)}$ of the selected complete paths is used as the reference value P_{fpm} for bounding (discarding) operations, *i.e.*,

$$P_{fpm} \triangleq \max P_{fp(q)(L)}{}^{(p_k)} \tag{18}$$

Branching operations are performed, based on the estimated upper bounds of the candidate failure paths. That is, the new failure path branches to the one which has the maximum value of $P_{fp(q)(U)}{}^{(p)}$. Further, the bounding (discarding) criterion is modified as

$$P_{fp(q)(U)}{}^{(p)} < 10^{-\delta} P_{fpm} \tag{19}$$

By using the approximations mentioned above, the resulting branch-and-bound algorithm gives the following properties.

Theorem 14 (Selected and discarded failure paths (2))

All the failure paths which may have the probabilities larger than/or equal to $10^{-\delta}P_{fpm}$ are selected. On the other hand, the discarded failure paths have the probabilities less than $10^{-\delta}P_{fpm}$.

Proof

From the bounding (discarding) criterion (19), the failure paths with the probabilities

$$P_{fp(q)}{}^{(p)} \leq P_{fp(q)(U)}{}^{(p)} < 10^{-\delta} P_{fpm} \tag{20}$$

are discarded, which proves the second part of the theorem. Then, the first part follows evidently.

Theroem 15 (The most probable failure path (2))

The probability of the most probable failure path is bounded as follows:

$$P_{fpm} = \max_q P_{fp(q)(L)}{}^{(p_k)} \leq \max_q P_{fp(q)}{}^{(p_k)} \leq \max_q P_{fp(q)(U)}{}^{(p_k)} \tag{21}$$

Proof

The probability $\max_q P_{fp(q)}{}^{(p_k)}$ of the most probable failure path is clearly bounded from below and above by the maximums of the lower and upper bounds of the selected complete failure path probabilities, respectively.

5. Concluding Remarks

The essential differences between the present and previous[11),13~15),22),23),26)] branch-and-bound procedures lie in the two points: First, the reference value P_{fpm} for the discarding operations is taken in the former as the maximum of the lower bounds of the selected complete failure path probabilities while it is taken as that of the upper bounds in the latter. That change assures the sound properties of the present method, as proved in the preceding section. Second, the branching after reaching the complete failure path is made in the present method to the most probable failure path among all the remaining paths. In the previous method, however, selection is performed first from the last failure stage if any paths left and then to the preceding failure stages.

The enumeration method is the special case of the proposed branch-and-bound method since by taking the bounding constant $\delta=+\infty$ all the complete failure paths are selected. The branch-and-bound method is computationally efficient when some failure paths are probabilistically dominant. On the other hand, its effectiveness is lost when all the failure paths are equally probable and then all of them are selected. For the latter case, an alternative method has to be developed, which hopefully enables us to evaluate the resulting errors.

Nothing has been mentioned of the mechanical properties of the structural systems. The treatment is essentially directed to the most general frame structures whose failure criteria, *i.e.*, formation of a collapse mechanism, are path-dependent. For the systems with path-independent failure criteria, *e.g.*, purely plastic-elastic frame structures, the branch-and-bound method can be computationally improved by restricting[26)] the partitioning failure paths because only one complete failure path needs to be selected for the same set of plastic hinges to yield system failure.

The system failure probability P_f is estimated by

$$P[\bigcup_{q\in X_c}(\bigcap_{i=1}^{p_k}F_{r_i(q)}^{(i)})]\le P_f\le P[\bigcup_{q\in X_c}(\bigcap_{i=1}^{p_k}F_{r_i(q)}^{(i)})\bigcup_{q\in X_t}(\bigcap_{i=1}^{p}F_{r_i(q)}^{(i)})]$$
$$\le P[\bigcup_{q\in X_c}(\bigcap_{i=1}^{p_k}F_{r_i(q)}^{(i)})]+E \tag{22}$$

where the union with respect to q means to be taken over all the selected complete failure paths X_c or all the discarded failure paths X_t and E is the contribution of the discarded failure paths. The probabilities of the unions of the intersections of the failure events are evaluated by using the well developed approximation methods[4~6),29)]. When the contribution E of the discarded failure paths is difficult to estimate or too large, an alternative upper bound $P_{fU}{}^{(p)}$ may be applied:

$$P_{fU}{}^{(p)}=P[\bigcup_q(\bigcap_{i=1}^{p}F_{r_i(q)}^{(i)})] \tag{23}$$

where the union with respect to q is taken over all the partial failure paths with length p. Eq. (23) evidently gives the upper bound due to the monotonicity of the failure path probabilities. Particularly when $p=1$, it becomes

$$P_{fU}{}^{(1)}=P[\bigcup_{i=1}^{n} F_{r_i}^{(1)}] \tag{24}$$

which is the weakest link approximation of the redundant structure.

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Appendix-Lower and upper bounds of failure path probability

A failure path probability is evaluated as

$$\begin{aligned} P_{fp(q)}{}^{(p)} &= P[\bigcap_{i=1}^{p} F_{r_i(q)}^{(i)}] = 1 - P[\bigcup_{i=1}^{p} \bar{F}_{r_i(q)}^{(i)}] \\ &\leq \min_{j\in\{2,3,\cdots,p\}} P[F_{r_1(q)}^{(1)} \cap F_{r_j(q)}^{(j)}] \triangleq P_{fp(q)(U)}{}^{(p)} \end{aligned}$$

where $\overline{(\,.\,)}$ is the complement of the failure event (.).
Consequently, $P_{fp(q)(U)}{}^{(p)}$ gives an upper bound.

The second term of the third expression is also evaluated as

$$\begin{aligned} &P[\bigcup_{i=1}^{p} \bar{F}_{r_i(q)}^{(i)}] \\ &= P[\bar{F}_{r_1(q)}] + P[F_{r_1(q)}^{(1)} \cap \bar{F}_{r_2(q)}^{(2)}] + \cdots + P[F_{r_1(q)}^{(1)} \cap F_{r_2(q)}^{(2)} \cap \cdots \cap F_{r_{p-1}(q)}^{(p-1)} \cap \bar{F}_{r_p(q)}^{(p)}] \\ &\leq 1 - P[\bar{F}_{r_1(q)}] + P[F_{r_1(q)}^{(1)} \cap \bar{F}_{r_2(q)}^{(2)}] + \sum_{j=3}^{p} \min\{P_{fp(q)(U)}{}^{(j-1)}, P[F_{r_1(q)}^{(1)} \cap \bar{F}_{r_j(q)}^{(j)}]\} \end{aligned}$$

Then, Eq. (17) follows.

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