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メタデータ	言語: eng
	出版者:
	公開日: 2010-04-06
	キーワード (Ja):
	キーワード (En):
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URL	https://doi.org/10.24729/00008590

On the Fourier coefficients of Eisenstein series for $\Gamma_0(N)$

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(Received June 15, 1983)

The Fourier coefficients of Eisenstein series of Nebentype for the congruence subgroup of level N are given explicitly, and the coefficients of Theta-series defined by some quadratic forms are determined using them.

§ 1. Introduction

Let $\mathbf{Z}$ denote a ring of rational integers, and $SL_2(\mathbf{Z})$ denote the elliptic modular group defined by

$$SL_2(\mathbf{Z})=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad-bc=1,\ a,\ b,\ c,\ d \in \mathbf{Z}\right\}.$$

Let $\Gamma(N)$ denote the principal congruence subgroup (of $SL_2(\mathbf{Z})$) of level N, i.e.

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid a \equiv d \equiv 1,\ b \equiv c \equiv 0 \bmod N\right\},$$

and $\Gamma_0(N)$ denote the congruence subgroup of level N, i.e.

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \mid c \equiv 0 \bmod N\right\}.$$

Let $\boldsymbol{C}$ denote the complex field, and $\boldsymbol{H}$ denote the complex upper-half plane i.e.

$$\boldsymbol{H}=\{z \in \boldsymbol{C} \mid \operatorname{Im}(z)>0\}.$$

Let k be an integer. A $\boldsymbol{C}$-valued function $f(\tau)$ on $\boldsymbol{H}$ is called a modulor form of weight k with respect to $\Gamma(N)$, if $f(\tau)$ satisfies the following three conditions:

(i) $f(\tau)$ is holomorphic on $\boldsymbol{H}$;

(ii) $f(\tau) \mid L=f\left(\frac{a\tau+b}{c\tau+d}\right)/(c\tau+d)^k=f(\tau)$

for all $L=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$;

(iii) $f(\tau)$ is holomolphic at every cusp of $\Gamma(N)$.

For any integer n prime to N, let $\boldsymbol{R}_n$ denote an element of $\Gamma(1)$ such that

$$\boldsymbol{R}_n=\begin{pmatrix} n^{-1} & 0 \\ 0 & n \end{pmatrix} \bmod N.$$

Let ε be a Dirichlet character defined mod N and $U=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Let $F(\tau)$ be a non-zero modular form of weight k with respect to $\Gamma(N)$ such that

$$F(\tau) \mid U=F(\tau) \text{ and } F(\tau) \mid \boldsymbol{R}_n=\varepsilon(n)F(\tau).$$

Since $\Gamma_0(N)$ is generated by U and $\boldsymbol{R}_n$ modulo $\Gamma(N)$, it is easy to see,

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$$(*)\quad F(\tau)\mid L=\varepsilon(d)F(\tau),\ \text{for all } L=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(N).$$

Further, $F\neq 0$ implies $\varepsilon(-1)=(-1)^k$, by above (ii).

$$\text{i.e. } \varepsilon(n)=\chi(n)=\left(\frac{n}{p_1}\right)^{r_1}\left(\frac{n}{p_2}\right)^{r_2}\dots\ (N=p_1^{r_1}p_2^{r_2}\dots),$$

where $(-)$ denotes the Legendre symbol.

The form $F(\tau)$ which satisfy (*) with $\chi(d)=\varepsilon(d)$ is called a modular form of *type*$(-k, N, \chi(d))$ (Nebentype). Furthermore we assume that $\chi(-1)=(-1)^k$.

If N is a prime p, Hecke ([3]) defined Eisenstein series of *type*$(-k, p, \chi(d))$ for $\Gamma_0(N)$ by the following $E_i(\tau)$ $(i=1, 2)$, using general Eisenstein series $G_k(\tau; a, b, N)$ for $\Gamma(N)$ (see below for the definition of $G_k(\tau; a, b, N)$),

$$2(-2\pi i)^k E_1(\tau)=(k-1)!\sum_{t,\,l \bmod p}\chi(t)G_k(\tau; t, l, p),$$

$$2(2\pi)^k E_2(\tau)=\gamma_k p^{k-1/2}(k-1)!\sum_{t \bmod p}\chi(t)G_k(\tau; 0, t, p).$$

And, their Fourier expansions with respect to $z=\exp(2\pi i\tau)$ are also obtained as follows.

$$E_1(\tau)=\sum_{n=1}^{\infty}c_1(n)z^n,\ c_1(n)=\sum_{d\mid n,\, d>0}d^{k-1}\chi(n/d),$$

$$E_2(\tau)=A_k(p)+\sum_{n=1}^{\infty}c_2(n)z^n,\ c_2(n)=\sum_{d\mid n,\, d>0}d^{k-1}\chi(d)$$

where

$$A_k(p)=\gamma_k p^{k-1/2}(k-1)!\,(2\pi)^{-k}L(k,\chi),\ L(k,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-k}$$

and $\gamma_k=(-1)^{[k/2]}$.

Let $Q(x)$ be the quadratic form i.e.

$$Q(x)=Q(x_1, x_2, \dots, x_f)=\sum_{1\leqq r\leqq s\leqq f}b_{rs}x_r x_s.$$

Then Theta-series $\vartheta(\tau, Q)$ is defined by the qudratic form $Q(x)$ as follows.

$$\vartheta(\tau, Q)=\sum_{m=0}^{\infty}a(m, Q)z^m,$$

where $a(m, Q)$ is the number of integral solutions $(x_1, x_2 \dots x_f)$ of the equation $m=Q(x_1, x_2, \dots, x_f)$, and $z=\exp(2\pi i\tau)$.

The purpose of this note is to construct Eisenstein series of *type*$(-k, N, \chi(d))$ for non-prime N and give the coefficients of their Fourier expansions explicitly. Furthermore, using these coefficients of Fourier expansions, we determine explicitly, in some numermerical examples, the coefficients $a(m, Q)$ of Thetaseries $\vartheta(\tau, Q)$.

§ 2. Fourier coefficients of Eisenstein series of Nebentype for $\Gamma_0(N)$

We shall review some results about $G(\tau; a_1, a_2, N)$ (Hecke [3]).

Let a_1, a_2 and k $(k\geqq 2)$ be rational integers. For $k\geqq 3$, let $G(\tau; a_1, a_2, N)$ denote the following series,

(1) $G_k(\tau;\, a_1,\, a_2,\, N)=\sum'_{\substack{m_1\equiv a_1\ (mod\ N)\\ m_2\equiv a_2\ (mod\ N)}} (m_1\tau+m_2)^{-k}$,

where the summation $\sum'$ runs through the pair of integers $(m_1, m_2)\neq(0, 0)$ and τ is complex variable with positive imaginary part.

Then their Fourier expansions at the cusp $i\infty$ of $\Gamma(N)$ is given as follows.

(2) $G_k(\tau;\, a_1,\, s_2,\, N)=\delta(a_1/N)\sum_{m_2\equiv a_2(N)} 1/m_2{}^k$

$(-2\pi i)^k N^{-k}/(k-1)!\sum_{\substack{mm_1>0\\ m_1\equiv a_1(N)}} m^{k-1}\, sgn(m)\, \zeta_N^{a_2m}\exp(2\pi imm\tau/N)$,

where

$$\delta(x)=\begin{cases}1 & \text{if } x \text{ is a rational integer}\\ 0 & \text{if } x \text{ is oterwise,}\end{cases}$$

$\zeta_N=\exp(2\pi i/N)$,

and

$sgn(x)=x/|x|$ for $x\neq 0$; $sgn(0)=0$.

If $k=2$, put

(3) $G_k(\tau;\, a_1,\, a_2,\, N)=-2\pi iN^{-2}/(\tau-\bar{\tau})$

$+\delta(a_1/N)\sum'_{m_2\equiv a_2(N)} 1/m_2{}^2-4\pi^2/N^2\sum_{\substack{mm_1>0\\ m_1\equiv a_1(N)}} |m|\,\zeta_N^{a_2m}\exp(2\pi imm_1\tau/N)$.

For every $k\geqq 2$, the series $G_k(\tau;\, a_1,\, a_2,\, N)$ satisfy the following properties (4) and (5).

If $a_1\equiv b_1$ and $a_2\equiv b_2$ (mod N), then

(4) $G_k(\tau;\, a_1,\, a_2,\, N)=G_k(\tau;\, b_1,\, b_2,\, N)$.

For each $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ in $SL_2(\boldsymbol{Z})$,

(5) $G_k\left(\frac{a\tau+b}{c\tau+d};\, a_1,\, a_2,\, N\right)/(c\tau+d)^k=G_k(\tau;\, aa_1+ca_2,\, ba_1+da_2,\, N)$.

Lemma 1. *Let Γ_∞ denote the subgroup of $\Gamma(1)$ generated by the matrix U. Then cardinality of $\Gamma_0(N)/\Gamma(1)/\Gamma_\infty$ is equal to $\sum_{t|n}\varphi(t, N/t))$ (here we denote φ by the Euler functon), and its cardinality is number of inequivalent classes of cusps of $\Gamma_0(N)$. Moreover each class of cusps of $\Gamma_0(N)$ is represented by a pair of coprime integers $\{x, r\}$, where r is a positive divisor of N and x is uniquely determined mod $(r, N/r)$.*

Proof. We refer to [1], [5] or [6] for the proof of this Lemma 1. Q.E.D.

Let $\{x, r\}$ be an inequivalent class of cusps of $\Gamma_0(N)$ in Lemma 1. For each class $\{x, r\}$, we define the Eisenstein series of *type*$(-k, N, \chi)$ for $\Gamma_0(N)$ as follows.

(6) $E_{\{x,r\}}(\tau)=\sum_{\substack{a\ mod\ N\\ b\ mod\ N}} \chi(a)G_k(\tau;\, rxa,\, rxb+a',\, N)$,

where a' is an integer uniquely determined *mod* N such that $aa'\equiv 1$ (N).

Theorem. *Let $E_{\{x,r\}}(\tau)$ be the series defined as above. Then the following statements hold:*

(i) *$E_{\{x,r\}}(\tau)$ are modular form of type$(-k, N, \chi)$.*

(ii) *Every modular form on type$(-k, N, \chi)$ is expressed as a linear combination of $E_{\{x,r\}}(\tau)$ and cusps forms.*

Proof. The statements of this Theorem are proved in a similar method given in [3]. Fully proof shall be given other places (We refer to [5] for the other proof in a general point of view). Q.E.D.

Now, let N be a product of two distinct odd primes p and q (i.e. $N=pq$). By Lemma 1, there are only four inequivalent class of represented by $\{0, 1\}$, $\{1, p\}$, $\{1, q\}$, $\{1, pq\}=i\infty$. We consider the normalised Eisenstein series $E^*_{\{x,r\}}(\tau)$ in place of $E_{\{x,r\}}(\tau)$. Then by (6) $E^*_{\{x,r\}}(\tau)$ are given as follows.

$$E_0(\tau)=E^*_{\{0,1\}}(\tau)=\frac{(k-1)!}{2(z\pi i)^k}\sum_{a \bmod (pq)}\chi(a)G_k(\tau; a, b, pq),$$

$$E_\infty(\tau)=E^*_{\{1,pq\}}(\tau)=\frac{\gamma_k(k-1)!(pq)^{k-½}}{2(-2\pi i)^k}\sum_{\substack{a \bmod (pq)\\ b \bmod (pq)}}\chi(a)G_k(\tau; 0, a, pq),$$

$$E_p(\tau)=E^*_{\{1,p\}}(\tau)=\frac{\tilde{\gamma}_p p^{k-3/2}(k-1)!}{2(-2\pi i)^k}\sum_{\substack{a \bmod (pq)\\ b \bmod (pq)}}\chi(a)G_k(\tau; pa, pb+a', pq),$$

$$E_q(\tau)=E^*_{\{1,q\}}(\tau)=\frac{\tilde{\gamma}_q q^{k-3/2}(k-1)!}{2(-2\pi i)^k}\sum_{\substack{a \bmod (pq)\\ b \bmod (pq)}}\chi(a)G_k(\tau; qa, qb+a', pq).$$

In following proposition, we give the Fourier coefficients of $E_0(\tau)$, $E_\infty(\tau)$, $E_p(\tau)$, and $E_q(\tau)$ explicitly.

Proposition 1. *Keep the notation as above. Then the Fourier expansions with respect to $z=exp(2\pi i\tau)$ of E_0, E_∞, E_p and E_q are given as follows.*

$$E_0(\tau)=\sum_{n=1}^{\infty} c_0(n)z^n,\ c_0(n)=\sum_{d|n, d>0} d^{k-1}\chi(n/d),$$

$$E_\infty(\tau)=A_k(pq)+\sum_{n=1}^{\infty} c_\infty(n)z^n,\ c_\infty(n)=\sum_{d|n, d>0} d^{k-1}\chi(d),$$

$$E_p(\tau)=\sum_{n=1}^{\infty} c_p(n)z^n,\ c_p=\sum_{d|n, d>0} d^{k-1}\chi_p(d)\chi_q(n/d),$$

and

$$E_q(\tau)=\sum_{n=1}^{\infty} c_q(n)z^n,\ c_p(n)=\sum_{d|n, d>0} d^{k-1}\chi_q(d)\chi_q(n/d)$$

where

$$\chi(n)=\chi_p(n)\chi_q(n),\ \chi_p(n)=\left(\frac{n}{p}\right),\ \chi_q(n)=\left(\frac{n}{q}\right) \text{ and}$$

$$A_k(pq)=\gamma_k(pq)_\chi^{-1/2}(k-1)!(2\pi)^{-k}L(k, \chi).$$

Proof. By (2) and (3), we see

$$\sum_{a \bmod (pq)} \chi(a) G_k(\tau; 0, a, pq) = \sum_{a \bmod (pq)} \chi(a) \sum_{m_2 = a} 1/m_2^k$$
$$+(-2\pi i)^k p^{-k} q^{-k}/(k-1)! \sum_{a \bmod (pq)} \chi(a) \sum_{\substack{mm_1>0 \\ m_1 \equiv 0 (pq)}} m^{k-1} sgn(m)\, \zeta_{pq}^{am} \exp(2\pi i m m_1 \tau/N).$$

The following resluts for Gauss sum is well known, i.e.

$$(7) \quad \sum_{a \bmod (pq)} \chi(a) \zeta_{pq}^{pm} = \tilde{\gamma}_{pq} \chi(m) (pq)^{1/2},$$

where, $\tilde{\gamma}_1$ deotes the number defined by

$$\tilde{\gamma}_l = \begin{cases} l & \text{if } l \equiv 1 \bmod 4 \\ i & \text{if } l \equiv 3 \bmod 4. \end{cases}$$

Using (7) we have

$$\sum_{a \bmod (pq)} \chi(a) \sum_{\substack{mm_1>0 \\ m_1 \equiv 0 (pq)}} ms^{k-1} gn(m)\, \zeta_{pq}^{am} \exp(2\pi i m m_1 \tau/pq)$$
$$= \sum_{\substack{mm_1>0 \\ m_1 \equiv 0 (pq)}} m^{k-1} sgn(m) \left(\sum_a \chi(a) \zeta_{pq}^{am}\right) \exp(2\pi i m m_1 \tau/pq)$$
$$= \tilde{\gamma}_{pq} (pq)^{1/2} \sum_{\substack{mm_1>0 \\ m_1 \equiv 0 (pq)}} m^{k-1} sgn(m) \chi(m) \exp(2\pi i m m_1 \tau/pq).$$

Futher we have

$$\sum_{\substack{mm_1>0 \\ m_1 \equiv 0 (pq)}} m^{k-1} sgn(m) \chi(m) \exp(2\pi i m m_1 \tau/pq)$$
$$= \sum_{m=1}^{\infty} m^{k-1} \chi(m) \{1 + \chi(-1)(-1)^k\} \sum_{m_1=1}^{\infty} \exp(2\pi i m m_1 \tau)$$
$$= 2 \sum_{m=1}^{\infty} \sum_{m_1=1}^{\infty} \chi(m) m^{k-1} \exp(2\pi i m m_1 \tau)$$
$$= 2 \sum_{d|n,\, d>0} d^{k-1} \chi(d) \exp(2\pi i n \tau).$$

Let $\gamma_k = (-1)^{[k/2]}$, then it is easy to see that $\gamma_k = i^k \tilde{\gamma}_{pq}^{-1}$.
Therefore we obtain $z(=\exp(2\pi i \tau))$-expansion of $E_\infty(\tau)$ given by

$$E_\infty(\tau) = \{p^{k-1/2} q^{k-1/2} (k-1)!\, \gamma_k\}/2(2\pi)^k \sum_a \chi(a) G_k(\tau; 0, a, pq)$$
$$= A_k(pq) + \sum_{n=1}^{\infty} \{\sum_{d|n,\, d>0} d^{k-1} \chi(d)\} z^n,$$

where

$$A_k(pq) = \{(pq)^{k-1/2} (k-1)!\, \gamma_k / 2(2\pi)^k\} \sum_a \chi(a) \sum_{m_2 = a (pq)} 1/m_2^k$$
$$= \{\gamma_k (pq)^{k-1/2} (k-1)!/(2\pi)^k\} L(k, \chi),$$

and

$$L(k, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-k}.$$

Once more, by (2) and (3) we see

$$\sum_{\substack{a \bmod pq\\ b \bmod pq}} \chi(a)G_k(\tau; pa, pb+a', pq)$$

$$=\frac{(-2\pi i)^k}{(pq)^k(k-1)!}\sum_{\substack{a \bmod pq\\ b \bmod pq}} \chi(a) \sum_{\substack{mm_1<0\\ m_1\equiv pa\,(pq)}} m^{k-1}\operatorname{sgn}(m)\,\zeta_{pq}^{(pb+a')m}\exp(2\pi i m m_1\tau/pq)$$

with $aa'\equiv 1 \bmod pq$.

$$\sum_{\substack{a \bmod pq\\ b \bmod pq}} \chi(a) \sum_{\substack{mm_1>0\\ m_1\equiv pa\,(pq)}} m^{k-1}\operatorname{sgn}(m)\,\zeta_{pq}^{(pb+a')m}\exp(2\pi i m m_1\tau/pq)$$

$$=\Big(\sum_{b=0}^{pq-1}\zeta_q^{bm}\Big)\sum_{a \bmod pq}\chi(a)\sum_{\substack{mm_1>0\\ m_1\equiv pa}} m^{k-1}\operatorname{sgn}(m)\,\zeta_{pq}^{a'm}\exp(2\pi i m m_1\tau/pq)$$

$$=q^k p\sum_{a \bmod pq}\chi(a)\sum_{\substack{mm_1>0\\ m_1\equiv pa}} m^{k-1}\operatorname{sgn}(m)\,\zeta_p^{a'm}\exp(2\pi i m m_1\tau/p)$$

$$=q^k p\sum_{a \bmod q}\sum_{mm_1>0}\chi_q(a)\Big(\sum_{\substack{c\equiv a\,(q)\\ (c,p)=1}}\chi_p(c)\,\zeta_p^{cm}\Big)m^{k-1}\operatorname{sgn}(m)\exp(2\pi i m m_1\tau/p)$$

$$=\tilde{\gamma}_p q^k p^{3/2}\sum_{a \bmod q}\sum_{\substack{mm_1>0\\ m_1\equiv pa}}\chi_p(m)\chi_q(a)m^{k-1}\operatorname{sgn}(m)\exp(2\pi i m m_1\tau/p)$$

$$=\tilde{\gamma}_p q^k p^{3/2}\sum_{m=1}^{\infty}\chi_p(m)m^{k-1}\sum_a\chi_q(a)\sum_{n=1}^{\infty}\{z^{(nq-q+a)m}+\chi_p(-1)(-1)^k Z^{(nq-a)m}\}$$

$$=2_p\tilde{\gamma} q^k p^{3/2}\sum_{m=1}\chi_p(m)m^{k-1}\sum_{n=1}^{\infty}\Big\{\sum_{a=1}^{q-1}\chi_q(nq-a)z^{(nq-a)m}\Big\}$$

$$=2\tilde{\gamma}_p q^k p^{3/2}\sum_{d\mid n,\, d>0} d^{k-1}\chi_p(d)\chi_q(n/d)z^n.$$

Here we are over proof of the statements for $E_\infty(\tau)$ and $E_p(\tau)$. It is easy to carry out them for $E_0(\tau)$ and $E_q(\tau)$. Q.E.D.

§ 3. Examples

In this section, we give some relations between Eisenstein series $E(\tau)$ and Theta-series $\vartheta(\tau, Q)$ defined by the quaduratic form $Q(x)$.

Let μ, $g_i(i=1, \dots, f)$ be integers such that $N=1+\mu\sum_{i=1}^{f} g_i$.

For such integers μ, g_i, we define a quadratic form $Q(x)=Q(x_1, \dots, x_f)$ and the Teta-series $\vartheta(\tau, Q)$ associated with the $Q(x)$ by

$$Q(x)=1/2\,\Big\{\sum_{i=1}^{f} x_i^2+\mu\Big(\sum_{i=1}^{f} g_i x_i\Big)^2\Big\},$$

and

$$\vartheta(\tau, Q)=\sum_{m=0} a(m, Q)z^m.$$

It is well known that the Theta-series $\vartheta(\tau, Q)$ is a modular form of *type*$(-f/2, N, \chi(d))$ (see Hecke).

Here we give three examples. In every case, if $\mu\fallingdotseq(N-1)/\sum_{i=1}^{f} g_i$ the coefficients

$a(m, Q)$ of Theta-series $\vartheta(\tau, Q)$ are given by the coefficients of Eisenstein series $E(\tau)$. But if $\mu=(N-1)/\sum_{i=1}^{f} g_i$, it is not given.

Example 1. $(f, N, k)=(4, 21, 2)$: We obtain two solutions $(\mu, g_1, g_2, g_3, g_4)=(5, 1, 1, 1, 1), (1, 3, 3, 1, 1)$ for the equation $1+\mu \sum_{i=1}^{4} g_i=21$, then $Q_1(x)$ and $Q_2(x)$ are defined by

$$Q_1(x)=1/2\{\sum_{i=1}^{4} x_i^2+5(\sum_{i=1}^{4} (x_i)^2\},$$

$$Q_2x=1/2\{\sum_{i=1}^{4} x_i^2+(x_1+x_2+3x_3+3x_4)^2\}.$$

Moreover, determining the number $a(n, Q_i)$ of integral solutions of the equation $Q_i(x_1, x_2, x_3, x_4)=n$, we obtain $\vartheta(\tau, Q_i)$ $(i=1, 2)$ i.e.

$$\vartheta(\tau, Q_1)=1+12z+6z^2+32z^3+36z^4+48z^5+\cdots,$$

$$\vartheta(\tau, Q_2)=1+8z+16z^2+30z^3+24z^4+72z^5+\cdots.$$

By Propositon 1, $E_0(\tau)$, $E_\infty(\tau)$, $E_3(\tau)$, and $E_7(\tau)$ are calculated as follows.

$$E_0(\tau)= \quad z+z^2+3z^3+3z^4+6z^5+\cdots,$$

$$E_\infty(\tau)=-1+z-z^2+z^3+3z^4+6z^5+\cdots,$$

$$E_3(\tau)= \quad z-z^2-z^3-3z^4-6z^5+\cdots,$$

and

$$E_7(\tau)= \quad z+z^2-3z^3+3z^4-6z^5+\cdots.$$

By the routive arguments, we have the relations of $\vartheta(\tau, Q_i)$ and $E(\tau)$, i.e.

$$\vartheta(\tau, Q_i)=21/2E_0(\tau)-1/2E_\infty(\tau)+7E_3(\tau)-3E_7(\tau),$$

$$\vartheta(\tau, Q_2)=21/2E_0(\tau)-1/2E_\infty(\tau)-7E_3(\tau)+3E_3(\tau).$$

Example 2. $(f, N, k)=(4, 69, 2)$: Carring similarly out the calculations as Lemma 1, we have each consequence as follows.

$$Q_1(x)=1/2\{\sum_{i=1}^{4} x_i^2+(5x_1+5x_1+3x_3+3x_4)^2\},$$

$$Q_2(x)=1/2\{\sum_{i=1}^{4} x_i^2+17(\sum_{i=1} x_i)^2\};$$

$$\vartheta(\tau, Q_1)=1+4z+8z^2+16z^3+12z^4+44z^5+\cdots,$$

$$\vartheta(\tau, Q_2)=1+12z+6z_2+24z^3+12z^4+24z^5+\cdots;$$

$$E_0(\tau)= \quad z+z^2+3z^3+3z^4+6z^5+\cdots,$$

$$E_\infty(\tau)=-12+z-z^2+z^3+3z^4+6z^5-\cdots,$$

$$E_3(\tau)= \quad z-z^2+z^3+3z^4-6z^5-\cdots,$$

$$E_{23}(\tau)= \quad z+z^2+3z^3+3z^4-6z^5+\cdots;$$

$$\vartheta(\tau, Q_1)=1/2\{69E_0(\tau)-E_\infty(\tau)-23E_3(\tau)+3E_{23}(\tau)\},$$

$$\vartheta(\tau, Q_2)=1/3S(\tau, Q_2)-1/2\{69E_0(\tau)-E_\infty(\tau)+23E_3(\tau)-3E_{23}(\tau),$$

$$S(\tau, Q_2)=4z+7z^2+17z^3-30z^4-20z^6-\cdots.$$

Example 3. $(f, N, k)=(4, 45, 2)$: In this case, N is not pq. By Lemma 1, there are only eight inequivalent class $\{0, 1\}$, $\{1, 45\}$, $\{1, 3\}$, $\{1, 5\}$, $\{1, 6\}$, $\{1, 9\}$, $\{1, 15\}$ and $\{1, 30\}$ of cusps for $\Gamma_0(45)$. Then we oftain the following Eisenstein series $E_0(\tau)$, $E_\infty(\tau)$, $E_3(\tau)$, $E_5(\tau)$, $E_6(\tau)$, $E_9(\tau)$, $E_{15}(\tau)$, $E_{30}(\tau)$.

$$E_0(\tau)= z+z^2+3z^3+3z^4+5z^5+3z^6+\cdots,$$
$$E_\infty(\tau)=-2+z^3-6z^6+7z^9+\cdots,$$
$$E_3(\tau)= z+\zeta_3{}^2z^2+3\zeta_3z^3+3z^4+5\zeta_3{}^2z^5+3\zeta_3z^6+\cdots,$$
$$E_5(\tau)= z-z^2-3z^3+3z^4+z^5+3z^6-\cdots,$$
$$E_6(\tau)= z+\zeta_3z^2+3\zeta_3{}^2z^3+3z^4+5\zeta_3z^5+3\zeta_3{}^2z^6+\cdots,$$
$$E_9(\tau)= z^3+z^6-7z^9+3z^{12}+\cdots,$$
$$E_{15}(\tau)=a(1)z-a(-1)z^2+3a(0)z^3+3a(1)z^4+a(-1)z^5-\cdots,$$
$$E_{30}(\tau)=a(-1)z-a(1)z^2+3a(0)z^3+3a(-1)z^4+a(1)z^5-\cdots,$$

where $a(0)=\sqrt{5}$, $a(1)=-\zeta_{15}^2-\zeta_{15}^8+\zeta_{15}^{11}+\zeta_{15}^{14}$,
$a(-1)=\zeta_{15}+\zeta_{15}^4-\zeta_{15}^7-\zeta_{15}^{13}$.
Let $E_3^*(\tau)$ and $E_{15}^*(\tau)$ denote as follows,

$$E_3^*(\tau)=-1/\sqrt{3}i\,\{\zeta_3{}^2E_3(\tau)-\zeta_3E_6(\tau)\},$$
$$E_{15}^*(\tau)=-1/\sqrt{15}i\,\{a(1)E_{15}(\tau)-a(-1)E_{30}(\tau)\},$$

then

$$E_3^*(\tau)=z-z^2+3z^4-5z^5+\cdots,$$
$$E_{15}^*(\tau)=z+z^2+3z^4-z^5-\cdots.$$

By two solutions $(\mu, g_1, g_2, g_3, g_4)=(11, 1, 1, 1, 1)$, $(1, 5, 3, 3, 1)$ of the equation $1+\mu\sum_{i=1}^4 g_i=45$, we define $Q_i(x)$ $(i=1, 2)$ as follows

$$Q_1(x)=1/2\,\{\sum_{i=1}^4 x_i+11(\sum_{i=1}^4 x_i)^2\},$$
$$Q_2(x)=1/2\,\{\sum_{i=1}^4 x_i+(5x_1+3x_2+3x_3+x_4)^2\}.$$

Then

$$\vartheta(\tau, Q_1)=1+12z+6z^2+24z^3+12z^4+24z^5+\cdots,$$
$$\vartheta(\tau, Q_2)=1+4z+2z^2+24z^3+12z^4+48z^5+\cdots.$$

By the routive arguments, we have the relations of $\vartheta(\tau, Q_i)$ and $E(\tau)$,

$$\vartheta(\tau, Q_2)=1/2\,\{15E_0(\tau)-E_\infty(\tau)-3E_5(\tau)-5E_9(\tau)-5E_3^*(\tau)+E_{15}^*(\tau)\},$$
$$\vartheta(\tau, Q_1)=S(\tau, Q_1)+1/2\,\{15E_0(\tau)-E_\infty(\tau)-3E_5(\tau)-5E_9(\tau)+5E_3^*(\tau)-E_{15}^*(\tau)\},$$
$$s(\tau, Q)=4z-12z^4-20z^{10}+\cdots.$$

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