An Approach to Reliability Evaluation of Redundant Structures

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The basic ideas of evaluating the reliability of structures are summarized. The effects of the type of structure and of probability distribution parameters are shown. Some methods of reliability analysis are compared together with their respective results when applied to certain sample structures. The idea of considering only the significant failure mode is introduced.

1. Introduction

As structural systems become more complex nowadays, the question of how safe or unsafe they are becomes increasingly important, too. The public wants to know the risks of failure of certain structures, the failure of which might lead to the loss of human life. On the other hand, the engineer might want to know how to design a structure or part thereof in order to maximize its overall reliability. For this, new techniques are necessary which model the structure's resistance on the one hand, and its loading on the other hand as random variables. Observed data show that loads and resistance are not deterministic values but have varying properties. Therefore the rules of stochastic theory have to be introduced into civil engineering practice. There are various publications on this matter, only references[16), 17)] may be mentioned here.

2. Reliability

2.1 General Remarks

In order to analyse the failure behavior of a structure, component failure has to be understood. In this paper only short notes can be given, and the chapter on component reliability may serve as an introduction to the basic ideas of reliability evaluation.

2.2 Component Reliability

A force S may be applied to a component with resistance R. This component fails if $R < S$, and survives if $R > S$. Since R and S may vary because of different building materials and load conditions, they should be defined as random variables with given or assumed mean, variance, and type of distribution. In this case the safety factor, *e.g.* the central safety factor, cannot be taken as measure of reliability any more. A better measure

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would be *e.g.* the probability of survival, or its complement, the probability of failure, which is given by

$$p_f = \mathrm{P}(R \leq S) = \mathrm{P}(R - S \leq 0) \tag{1}$$

In order to evaluate this failure probability the so-called convolution integral

$$p_f = \int_0^\infty F_R(x)\, f_S(x)\, dx \tag{2}$$

has to be introduced, where $F_R(x)$: cumulative distribution function (*c.d.f.*) of the resistance, and $f_S(x)$: probability density function (*p.d.f.*) of the load. It can be derived as follows: The probability of a load $S = x$ to occur within an interval dx is $f_S(x)\,dx$. The probability of R being smaller than this value is $F_R(x)$. Since R and S are usually assumed to be statistically independent, these values can be multiplied to give the probability that R is smaller than a certain occuring load $S = x$. To include the probability that any value of S may occur, the above integral has to be evaluated (see Fig. 1). By introducing the safety

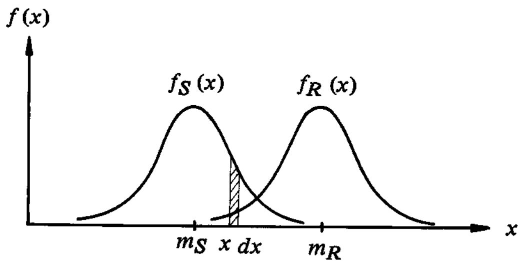

Fig. 1 Convolution

margin $Z = R - S$ as a new random variable, the probability of failure can also be defined by

$$p_f = \mathrm{P}(Z \leq 0) \tag{3}$$

In case Z is normally distributed Eq. (3) can directly be evaluated from the first two moments of Z – mean m_Z and standard deviation σ_Z – by using normal tables. If Φ denotes the standardized normal (Gaussian) distribution function, the failure probability is

$$p_f = \Phi\left(-\frac{m_Z}{\sigma_Z}\right) = \Phi(-\beta) \tag{4}$$

where β is the generalized safety index. If R and S are normally distributed, Eq. (4) is exact.

2.3 Reliability of Structures

2.3.1 Introduction

The influence of the behavior of just one component on the structure depends *e.g.* on the type of structure, on the material used, and on the considered failure criterion. Failure of a system might be assumed if any of the following criteria is applied:

- A certain limit stress is reached or exceeded at any one section, or a maximum deformation might be met at a certain point of the structure (elastic limit load).
- At least one component fails (plastification of section).
- The structure collapses partially or totally.

According to how "failure" is defined, there are of course different probabilities of this event. For example the elastic limit load of a simple one bay-one storey frame may occur with $p_f = 2.3 \times 10^{-1}$. This criterion can be applied to both determinate and indeterminate structures and usually implies collapse of a structure of brittle material. The second criterion would be the complete plastification of a section, which leads to the collapse of a determinate structure. The chain model (weakest link model) represents this criterion, which means that the system fails if any one member fails. For the given frame, the probability of this event to occur is $p_f = 8 \times 10^{-2}$. The third criterion – the plastic limit load – is applicable to redundant structures of plastic material. The structure will collapse only if a plastic hinge mechanism develops, *i.e.* if all redundant elements fail. A suitable model is that of a parallel system. The probability of the above frame to collapse is $p_f = 2 \times 10^{-4}$.

Of course, most realistic structures have to be described by a combination of the chain and the parallel model, *e.g.*, there are at least three different collapse mechanisms of the already mentioned frame each of which is represented by a parallel model; but as the structure can fail by any one mechanism, the appropriate overall model is the chain model.

According to how "failure" is defined (in the above sense), there are different "failure modes": If *e.g.* the elastic limit load is the applicable failure criterion, then "failure of component no. i" is one possible mode. If the plastic limit load is the criterion, then the formation of any plastic hinge mechanism is a failure mode. Of course, there are more possible definitions.

If the collapse failure probability of a statically indeterminate steel structure has to be evaluated, the plastic hinge mechanisms may be the modes to be considered. The performance functions or limit state functions of such plastic hinge mechanisms can be found by using the virtual work theorem. The general form of such an equation is given by

$$Z_i = \sum_k a_{ik} M_k + \sum_j b_{ij} L_j \tag{5}$$

Z_i : random variable describing the residual strength of the ith failure mode
a_{ik} : resistance coefficient at the kth point of the structure
M_k : random variable denoting plastic moment capacity at kth point of structure
b_{ij} : load coefficient related to jth load
L_j : random variable denoting jth load

When using this modified definition of the safety margin Z_i, system failure occurs if any $Z_i \leq 0$, and the respective failure probability is

$$p_f = \mathrm{P}(F_1 \cup F_2 \cup \ldots \cup F_i \cup \ldots \cup F_m) \tag{6}$$

To evaluate Eq. (6) this can be transformed in different ways, *e.g.*

$$p_f = \sum_{i=1}^{m} \mathrm{P}(F_i) - \sum_{i}\sum_{<j} \mathrm{P}(F_i F_j) + \sum_{i}\sum_{<j}\sum_{<k} \mathrm{p}(F_i F_j F_k) - \cdots \tag{7}$$

The intersecting probabilities lead to multiple integrals which usually are not easily evaluated. But there are two simple cases – when there is independence among the modes, and when there is perfect correlation respectively. In case of perfect correlation Eq. (7) reduces to

$$p_f = \max_{1 \leq i \leq m} p_{f,\,i} \tag{8}$$

and in case of independence $\mathrm{P}(F_i F_j) = \mathrm{P}(F_i) \times \mathrm{P}(F_j)$ which is negligible because of higher order, thus

$$p_f \leqq \sum_{i=1}^{m} p_{f,\,i} \tag{9}$$

Eqs. (8) and (9) form well-known lower and upper bounds to system failure probability. These bounds can be close if few modes are present or if one mode is dominant, and can be wider if the failure probabilities of several uncorrelated modes have the same magnitude.

2.3.2 Some Methods to Determine System Failure Probability

In case these bounds are not satisfying some methods are available to approximately calculate system failure probabilities:

From the statistical formulation for the probability of survival

$$p_S = \mathrm{P}(S_1 \cap S_2 \cap \ldots \cap S_i \cap \ldots \cap S_m) \tag{10}$$

Jorgenson and Goldberg[1] derive a multiple integral to calculate the system probability of survival. However, the necessary integration procedure is quite time consuming.

The complementary formulation to Eq. (10) is given by Eq. (6), which can be ex panded to a sum of mutually exclusive events

$$p_f = \mathrm{P}(F_1) + \mathrm{P}(S_1 F_2) + \mathrm{P}(S_1 S_2 F_3) + \ldots \tag{11}$$

Using this formulation, several authors try to take the correlation of modes into account. The various methods then introduce different simplifications. Stevenson and Moses[2] *e.g.* assume conditioned probabilities to be Gaussian, reduce multiple integrals by replacing random variables by their means, and limit conditioning events to three. Vanmarcke[3] reduces intersecting events to two, the correlation of which is approximately taken into account. Murotsu *et al.*[4] reduce the number of considered failure modes according to their contribution to system failure probability, and reduce high-dimensional joint probabilities. An approximating technique to include mode correlation between any two modes is also used by Ditlevsen[5], who introduced closer bounds. Basis of this method is a second-moment reliability format for which the basic variables have to be transformed into standardized, normally distributed variables. A "probabilistic network evaluation technique" is introduced by Ang and Ma[6]. It groups the modes according to their mutual correlation and avoids the difficult evaluation of intersecting events. All methods described so far need the evaluation of all modal limit state functions. A procedure which avoids this is proposed by Klingmüller[7]. From the theorem of total probability he derives the approximation that the system probability of failure is the failure probability of the deterministically relevant mechanism divided by the probability that this is the relevant mechanism. A plastic limit load analysis is also used by Kappler[8] in a failure tree analysis to find the dominant failure modes. System failure probability is assumed to be the probability of the last hinge to occur that is necessary to form a mechanism.

2.4 Influence of Parameters

Coefficient of Variation. A variation of distribution parameters can influence the value of failure probability. The most important in this respect is the coefficient of variation (*C.O.V.*) $V = \sigma/m$, which can be interpreted as relative standard deviation. Fig. 2 shows schematically how a larger *C.O.V.* causes larger failure probabilities: In the area

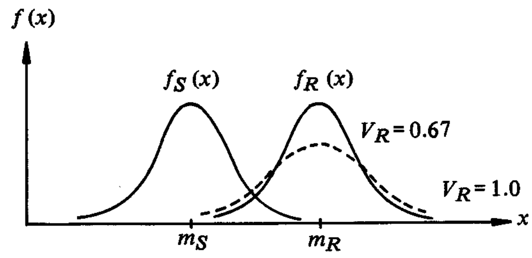

Fig. 2 Influence of *C.O.V.*

where the *p.d.f.* of the load has significant values, the probability density of the resistance with larger *C.O.V.* is larger by which the product in Eq. (2) is influenced. Fig. 3 shows this effect on the failure probabilities of sample structures of a benchmark study (see section 3).

Type of Distribution. The type of a distribution may significantly affect the value of failure probability – *e.g.* the normal distribution is symmetrical with respect to the mean, the lognormal is skewed. The effect of skewness as compared to a normal distribution is shown in Fig. 4 – again the *p.d.f.* of the resistance R has larger values in the tail region. The analysis of the same structures as before shows that the "lognormal" case leads to larger failure probabilities than the "normal" case, see Fig. 5.

Correlation of Basic Variables. The assumption of perfect correlation of the basic variables (*e.g.* plastic moment capacities) leads to larger system failure probabilities as compared to perfect independence. This can easily be seen from the variance of a sum of random variables $A = B + C$

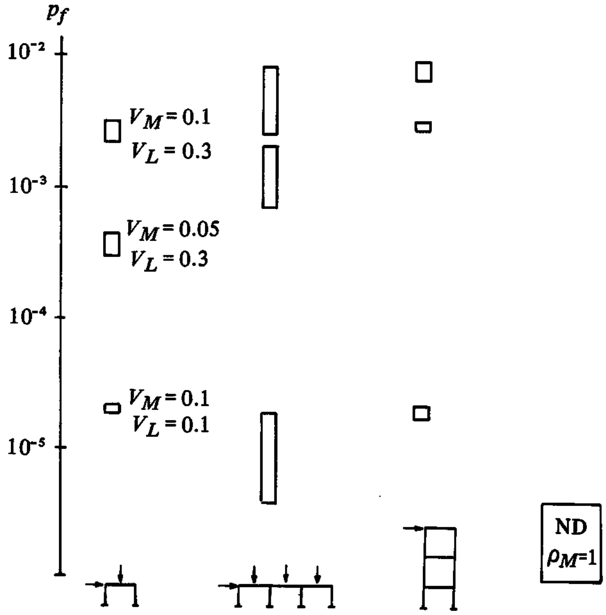

Fig. 3 Influence of C.O.V. on failure probability represented by general bounds (Eqs. (8), (9)) (ND : Normal distribution)

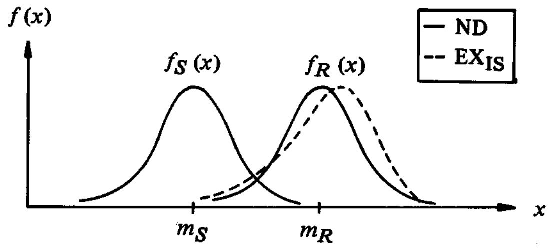

Fig. 4 Influence of type of distribution
ND : Normal distribution
EX_{IS} : Type I extreme value distribution of smallest values

$$\sigma_A{}^2 = \sigma_B{}^2 + 2 \times \rho_{BC} \times \sigma_B \times \sigma_C + \sigma_C{}^2 \tag{12}$$

with ρ_{BC} = correlation coefficient of $B, C,$ *i.e.* higher correlation (*e.g.* $\rho_{BC} \to 1$) leads to a larger variance and a larger failure probability. Fig. 6 shows this effect on the failure probabilities of the sample structures.

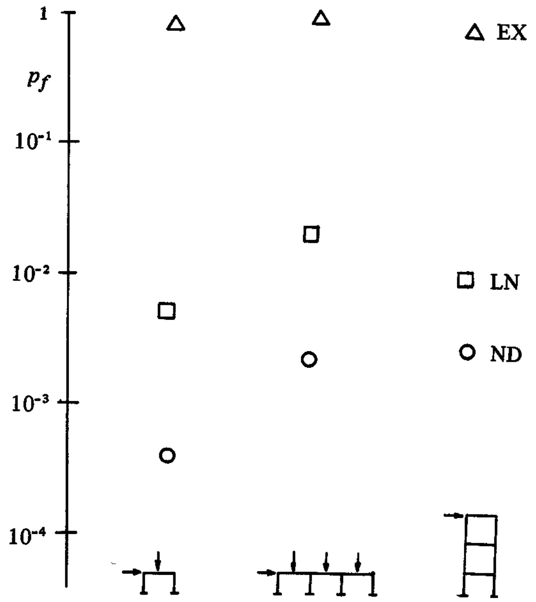

Fig. 5 Influence of type of distribution on failure probability represented by results of simulation ($V_M = 0.1$, $V_L = 0.3$, $\rho_M = 0$).
EX : Extreme value distribution LN : Log-normal distribution
ND : Normal distribution

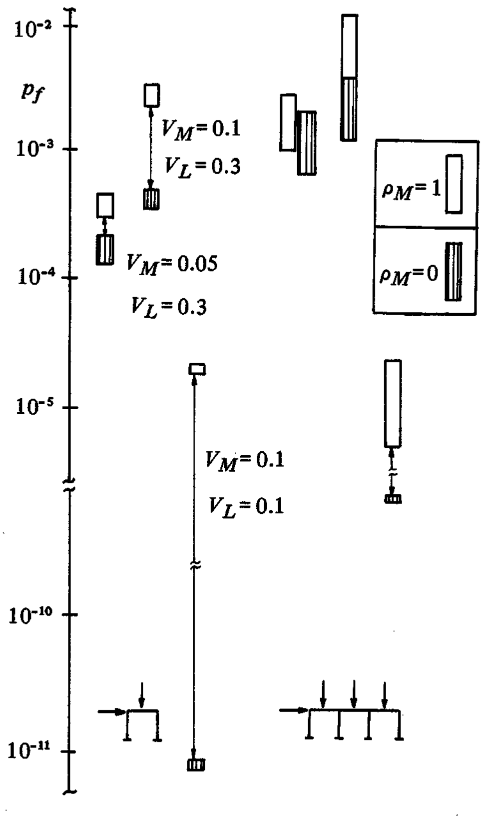

Fig. 6 Influence of correlation of plastic moment capacities on failure probability represented by general bounds (Eq. (8), (9)) : Normal distribution

3. Benchmark Study on Methods to Determine Collapse Failure Probabilities of Redundant Structures

3.1 Introduction

In the past a number of methods for determining collapse probabilities of redundant structures have been developed by various authors. Since these methods are based on different assumptions and techniques, the authors[9)] thought it necessary to evaluate their capabilities for various kinds of applications. For this reason a benchmark study was initiated, where structures and all other parameters were the same for all methods under consideration. The purpose for this comparison was to determine

- whether the methods yield different results when applied to the same structures and fixed parameters,
- which results they yield when applied to different types of structures,
- which limitations exist for the application of a certain method, and
- how difficult and time consuming the application is, *i.e.* how efficient a method is for particular cases.

Different types of structures showing different basic characteristics are to be evaluated (Fig. 7):

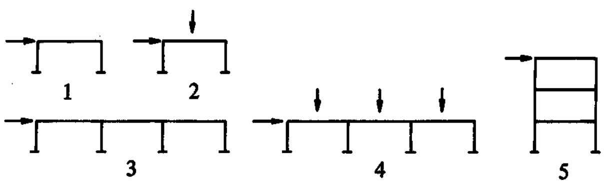

Fig. 7 Structures to be analysed.

Since only redundant structures of plastic material are to be considered, the applicable failure criterion in this study is the plastic limit load, *i.e.* the development of a plastic-hinge mechanism that leads to partial or total collapse of the structure. Neither the elastic limit load nor buckling or any other failure mode is to be considered. Only loads and plastic moment capacities are assumed to be random variables which are modeled by three different kinds of distribution. The cases of perfect correlation and no correlation among plastic moments are to be considered; the correlation of mechanisms (= modes) depends on the capability of the particular methods. They were described in section 2. 3. A Monte-Carlo-Simulation was also applied to the sample structures in order to verify the results obtained by the other methods.

3.2 Results

The study is based on results which are contributions of the respective authors themselves. They are compared and shortly summarized. A more detailed report on this study

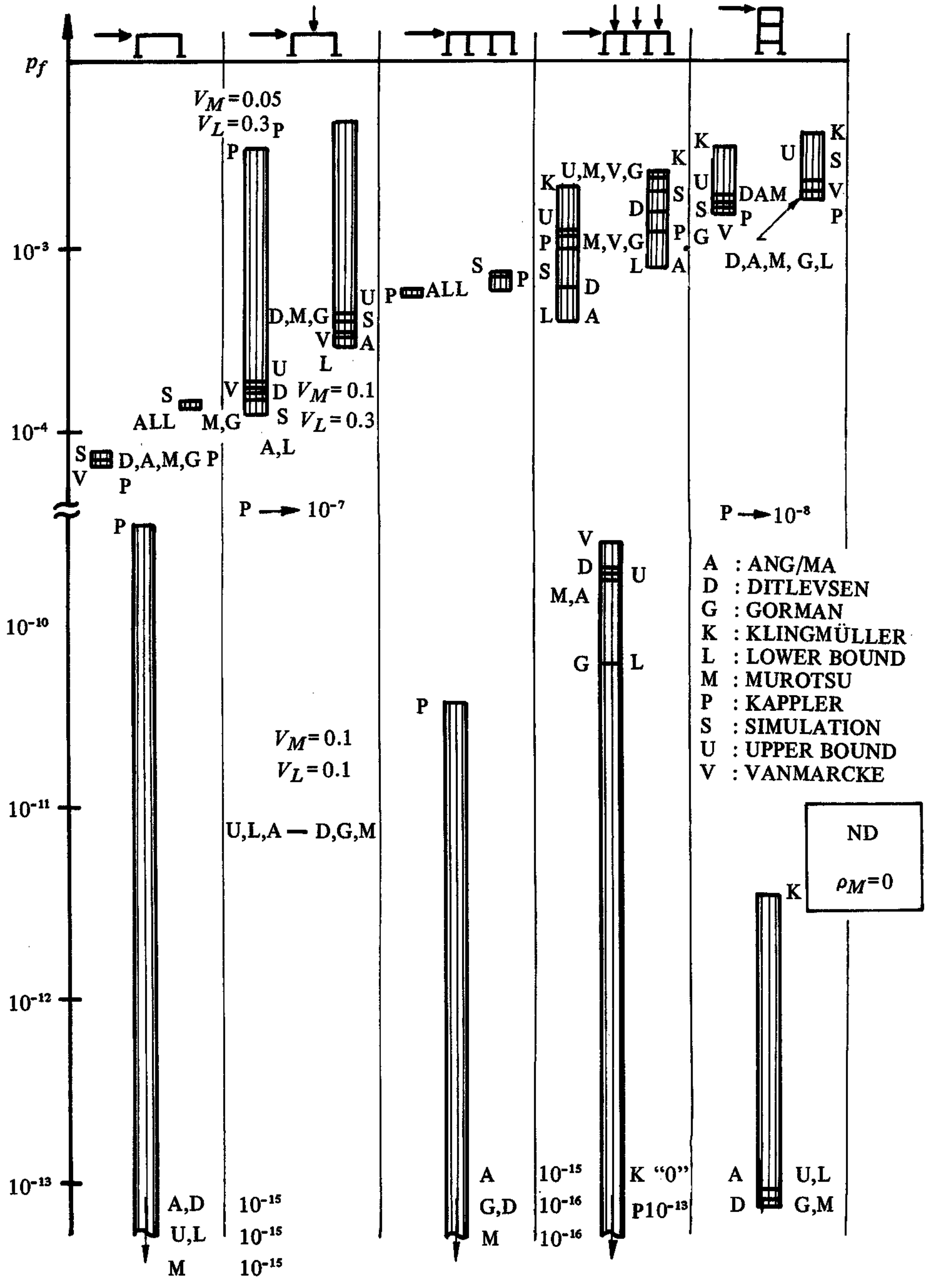

Fig. 8 Failure probabilities of various methods for various combinations of *C.O.V.* and different systems; Normal distribution, $\rho_M = 0$.

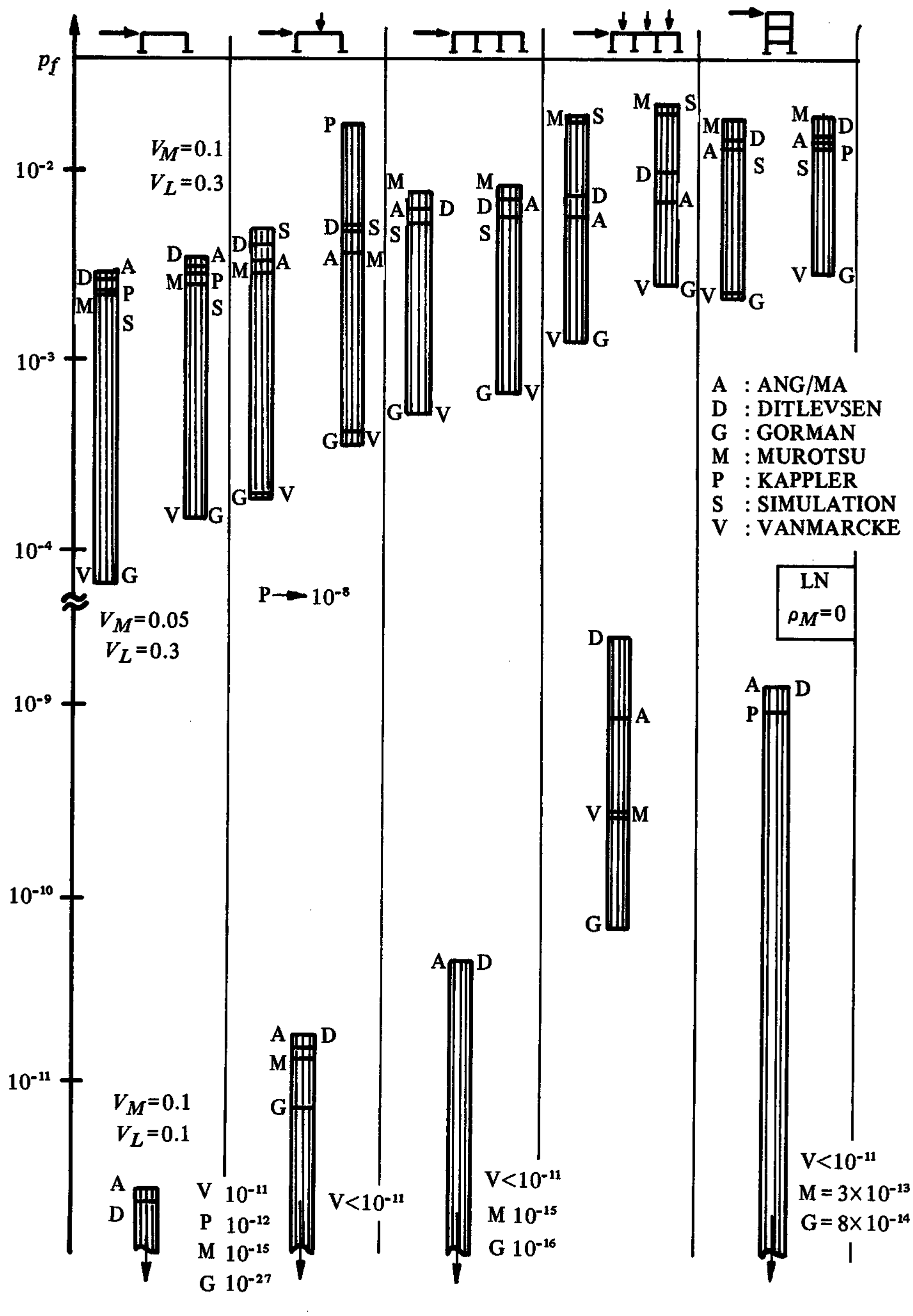

Fig. 9 Failure probabilities of various methods for various combinations of C.O.V. and different systems; Log-normal distribution, $\rho_M = 0$.

will be published[9].

The general influence of different parameters, *e.g.* coefficient of variation, type of distribution, correlation of basic variables, is well known and to be found in this study, too. As was expected the various methods do not yield identical results in all cases. As all of them are based on the analytically tractable normal distribution, in this case the failure probabilities according to the various methods do not differ significantly no matter which coefficient of variation and which plastic moment correlation is assumed (with the exception of one method), as can be seen from Fig. 8. But as soon as non-normal distributions are assumed, which in many cases is more realistic, the discrepancies of the results of the various methods may be considerably large, as Fig. 9 shows for lognormally distributed basic variables. Especially with small coefficients of variation these discrepancies might reach several orders of magnitude. This means, in case of non-normal distributions only those methods should be used that are capable to evaluate non-normal safety margins.

The common drawback of most methods is the fact that the modal performance functions have to be derived by hand, which – particularly with large systems – implies a considerable amount of work. Quite easily a significant failure mechanism might be missed. Those methods that need to evaluate only one failure mode, based on a plastic limit load analysis at mean values, which might not yield the stochastically most relevant mechanism. For these reasons, in order to be able to evaluate failure probabilities of large structures, a method has to be found which automatically yields the dominant failure modes.

4. Methods to Find Dominant Modes

If the failure probabilities of all possible modes can be evaluated, of course, the dominant modes are easily selected. The modes of a small system can be found by simple investigation, but this may be very tedious or impossible at all for large systems. For this reason an automatic method to find all modes would be very useful. This can be done by combining the basic independent mechanisms which can be found by matrix methods.[10] The dominant failure modes can be found, *e.g.* by searching for the minima of the generalized safety index β.[11]

A more interesting method is a branch-and-bound algorithm[12), 13] by which the significant failure modes can be found directly. By this, system failure is modeled by a sequence of members failing – the structure thus undergoing consecutive failure stages or failure levels. There is a large number of these possible failure paths all of which cannot be considered. So the number of the failure paths has to be reduced. This is done by deciding at every stage which member will most probably fail next, and by discarding possible failure paths branching from this level. The method was first applied to truss structures, but with some modifications is applicable to frame structures,[14), 15)] too.

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