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A Study of $\frac{1}{3}$ -harmonics in the neighborhood of Branching Point in Non-autonomous Piecewise Linear Systems with Unsymmetrical Restoring Force

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This is a study of $\frac{1}{3}$ -harmonics in the neighborhood of branching point in the non-autonomous second order differential equation with piecewise linear restoring force having unsymmetrical characteristics in the case of the undamped systems.

In this report the periodic condition in which the first variational equation has a periodic solution of period 3T (T : period of external force) and the branching behavior of the trajectories of $\frac{1}{3}$ -harmonics from that of harmonics are obtained.

1. Introduction

It is well known that nonlinear systems can possess a wide variety of periodic solutions in addition to those which have the same period as the external force[1), 2), 3), 4)]; for example, what are called subharmonic oscillations can occur in which the smallest period of the motion may be any integral multiple of the period of the external force. Such oscillations may occur in nonlinear systems, particularly in the case where the system is described by Duffing's equation with symmetrical restoring force.

In the preceding articles[5), 6), 7)], we have obtained some results as to periodic solutions in the nonautonomous piecewise linear systems with unsymmetrical restoring force.

We are now to deal with the subharmonic oscillation whose fundamental frequency is a fraction $\frac{1}{3}$ of the driving frequency in the neighborhood of the branching point, at which the subharmonic and the harmonic oscillations are identical, in piecewise linear systems with unsymmetrical restoring force, which has been studied very little.

This paper will qualitatively and numerically give the branching condition and some results as to the behavior of solutions near bifurcation point.

2. Periodicity Conditions

In this chapter we restate the periodicity conditions as to the solutions type ${}_1A$ shown in Fig. 2.

Consider the equation

$$\ddot{x} + f(x) = E \cos \omega t \tag{1}$$

where $f(x)$ is a piecewise linear restoring force (shown in Fig. 1) given by

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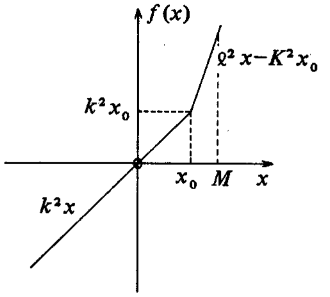

Fig. 1. Restoring force characteristics

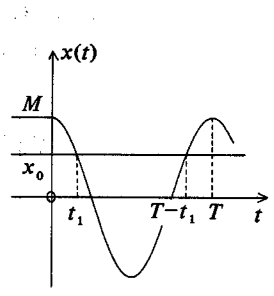

Fig. 2. Periodic solution of type $_1$A

$$f(x) = \begin{cases} \ell^2 x - K^2 x_0 & (x \geqq x_0) , \\ k^2 x & (x \leqq x_0) , \end{cases} \tag{2}$$
$$\ell^2 = k^2 + K^2$$

in which k, ℓ, K, and x_0 are positive constants.
In this paper dots over a quantity refer to differentiation with respect to the time.

Assume the initial conditions as follow,

$$x(0) = M (> x_0) , \quad \dot{x}(0) = 0 \tag{3}$$

Then, the conditions of periodicity are

$$\left(M - \frac{E}{\ell^2 - \omega^2} - \frac{K^2}{\ell^2} x_0 \right) \cos \ell t_1 + \frac{E}{\ell^2 - \omega^2} \cos \omega t_1 = \frac{k^2}{\ell^2} x_0 \tag{4}$$

$$\tan k \left(\frac{T}{2} - t_1 \right) = \frac{-\ell \left(M - \frac{E}{\ell^2 - \omega^2} - \frac{K^2}{\ell^2} x_0 \right) \sin \ell t_1 + \frac{\omega K^2 E}{(\ell^2 - \omega^2)(k^2 - \omega^2)} \sin \omega t_1}{k \left(x_0 - \frac{E}{k^2 - \omega^2} \cos \omega t_1 \right)} \tag{5}$$

where $T = 2\pi/\omega$ and t_1 is the time when the solution reaches the corner point first. Thus periodicity conditions depend on four parameters M, E, ω, and t_1 and so we have the parameters M and E which lead to periodic solutions if angular frequency ω and transition time t_1 are given, viz ;

$$M = M(\omega, t_1) , \quad E = E(\omega, t_1) \tag{6}$$

As to any other type of solutions, periodicity conditions can be obtained in the same procedure as for that of type $_1$A.

3. Branching Condition

We assume that equation (1) admits periodic solution $x^0(t)$ of period $T = 2\pi/\omega$ under the conditions (7) :

$$x^0(0) = M_0\,, \quad \dot{x}^0(0) = 0\,, \quad E = E_0 \tag{7}$$

It is of course clear that we have

$$\left.\begin{aligned} &x^0\,(t+T) = x^0\,(t)\;, \\ &x^0\,(t) = x^0\,(-t) \end{aligned}\right\} \tag{8}$$

The variational equation of equation (1) associated with the periodic solution $x^0(t)$ for the variation $y\,(t)$ is

$$\ddot{y} + a\,(t)\,y = 0 \tag{9}$$

where $a\,(t)$ is an even and T-periodic function which is given by the formula (shown in Fig. 3)

$$a\,(t) \equiv \left.\frac{\partial f(x)}{\partial x}\right|_{x = x^0(t)} = \begin{cases} \ell^2 \;(x^0\,(t) > x_0) \\ k^2 \;(x^0\,(t) < x_0) \end{cases} \tag{10}$$

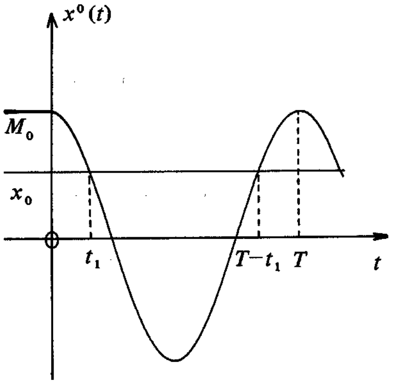

Fig. 3. Periodic solution x^0 (t)

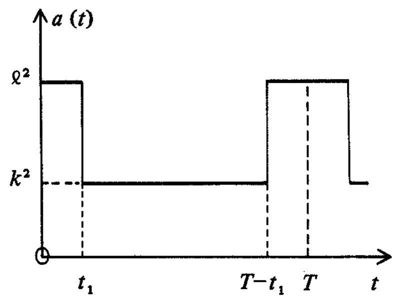

Fig. 4. Coefficient a (t) of periodic solution type $_1$A

Therefore equation (9) means a Hill's equation. It is known from Floquet theory as to normal solutions that

$$y\,(t+T) = \rho\, y\,(t) \tag{11}$$

and for

$$\rho = e^{\,j\frac{2m\pi}{n}} \quad (m,\, n \text{ relatively prime integers}) \tag{12}$$

there exist periodic solutions of period $n\,T$.

Here we shall investigate the case for $n = 3$, since we have already reported[5), 6)] in the case of $n = 1$ and $n = 2$. The condition $\rho = e^{\,j\frac{2m\pi}{3}}$ means that $y\,(t + 3T) = y\,(t)$,

$$\rho_1 \cdot \rho_2 = 1\;, \text{and}\;\; \rho_1 + \rho_2 = -1 \tag{13}$$

if ρ_1 and ρ_2 are the roots of the characteristic equation of equation (9).

Let $\varphi\,(t)$ and $\psi\,(t)$ be fundamental solutions of equation (9) which satisfy the following conditons at $t = 0$:

$$\left.\begin{array}{ll} \varphi(0) = 1 \ , & \dot{\varphi}(0) = 0 \\ \psi(0) = 0 \ , & \dot{\psi}(0) = 1 \end{array}\right\} \tag{14}$$

Then $\varphi(t)$ is an even function and $\psi(t)$ is an odd function. Using ρ_1 and ρ_2 expressed in terms of $\varphi(t)$ and $\psi(t)$, we have

$$\rho_1 + \rho_2 = \varphi(T) + \dot{\psi}(T) = -1 \tag{15}$$

and from the explicit formulas for $\varphi(t)$ and $\psi(t)$

$$\varphi(T) = \dot{\psi}(T) = \cos 2\ell t_1 \ \cos 2k\left(\frac{T}{2} - t_1\right) - \frac{1}{2}\left(\frac{k}{\ell} + \frac{\ell}{k}\right) \sin 2\ell t_1 \ \sin 2k\left(\frac{T}{2} - t_1\right) = -\frac{1}{2} \tag{16}$$

Therefore the solutions of period $3T$ of equation (9) are admitted when equation (16) holds and in this case the bifurcation of trajectories may occur.

Thus the branching condition is given by equation (16).

4. Branching Phenomena

In this chapter we shall discuss the branching of the trajectories of periodic solutions whose period are T and $3T$.

It is well known that bifurcation theory concerns the solution $x(t;\lambda)$ of a problem which depends upon a scalar or vector parameter λ[8]. The solution is said to bifurcate from the solution $x^0(t;\lambda_0)$ at the parameter value $\lambda = \lambda_0$ if there are two or more distinct solutions which approach $x^0(t;\lambda_0)$ as λ tends to λ_0.

The first problem of bifurcation theory is to determine the solution $x^0(t;\lambda_0)$ and parameter value λ_0 at which bifurcation occurs (see chapter 2, 3). The second problem is to find the number of solutions which bifurcate from $x^0(t;\lambda_0)$. A third problem is to determine the behavior of these solutions for λ near λ_0. The behavior of the solutions for λ outside a small neighborhood of λ_0 is also important, but is not considered in bifurcation theory.

Now we shall consider the branching phenomena of the curves which bifurcate the solutions of period $3T$ from the T-periodic solutions of equation (1).

Assume the solution x of equation (1) as a function of the three parameters, which has $x = M$, $\dot{x} = N$ at $t = 0$,

$$x = x(t; M, N, E) \tag{17}$$

Let the functions $F(M, N, E)$ and $G(M, N, E)$ be defined by the formulas :

$$\left.\begin{array}{l} F(M, N, E) \equiv x(3T; M, N, E) - x(0; M, N, E) \\ G(M, N, E) \equiv \dot{x}(3T; M, N, E) - \dot{x}(0; M, N, E) \end{array}\right\} \tag{18}$$

The solution $x(t; M, N, E)$ will be $3T$-periodic if and only if

$$F(M, N, E) = G(M, N, E) = 0 \tag{19}$$

The set points in (M, N, E)-space for which $F = G = 0$ will usually be composed of certain

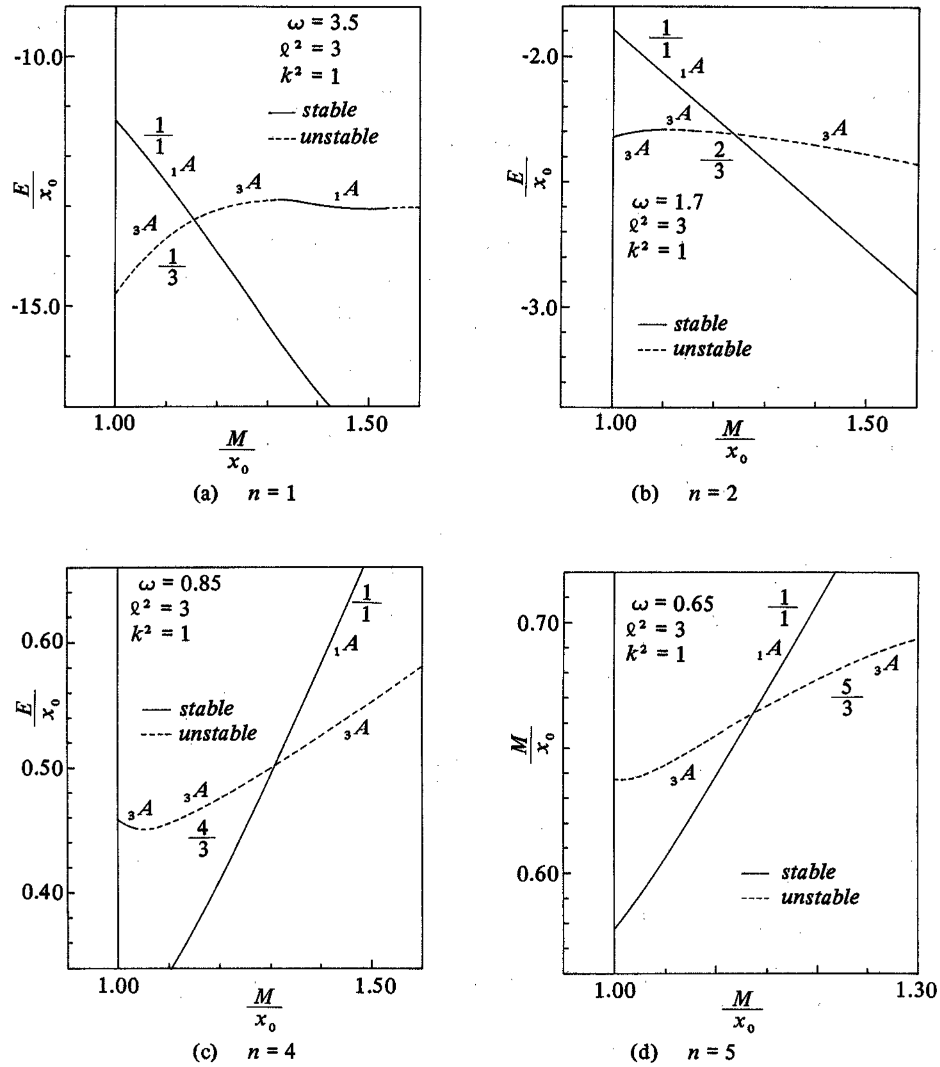

Fig. 5. Branching phenomena of solutions of order $\frac{n}{3}$ (n = 1, 2, 4, 5) from harmonic solutions

curves. To study the locus of $F(M, N, E) = G(M, N, E) = 0$ in the neighborhood of the point $(M_0, 0, E_0)$, satisfied by equations (7) and (16), we introduce the following notations.

Let $x^0(t)$ denote the T-periodic solution of equation (1). Let $\varphi(t)$ and $\psi(t)$ denote the solution of the variation equation (9).

We shall be evaluating the partial derivatives of F and G at the point $(M_0, 0, E_0)$. Let $x_E(t)$ denote the partial derivative of $x(t; M, N, E)$ with respect to E evaluated for $M = M_0, N = 0, E = E_0$.

Then $x_E(t)$ is the solution of the problem :

$$\left.\begin{array}{l}\ddot{y} + a(t)\,y = \cos\omega t,\\ y(0) = \dot{y}(0) = 0\end{array}\right\} \tag{20}$$

The first partial derivatives of F and G at the point $(M_0, 0, E_0)$ have the values :

$$\begin{array}{lll} F_M = \varphi(3T) - 1 = 0, & F_N = \psi(3T) = 0, & F_E = x_E(3T) = 0\\ G_M = \dot{\varphi}(3T) = 0, & G_N = \dot{\psi}(3T) - 1 = 0, & G_E = \dot{x}_E(3T) = 0\end{array} \tag{21}$$

The first partial derivatives of F and G at $(M_0, 0, E_0)$ both vanish.

The second derivatives of F and G are computed using the values of the second partial derivatives of $x\,(t;\,M,\,N,\,E)$ at $(M_0,\,0,\,E_0)$, but we shall omit most of them in what follows, since these computations are long and tedious.

It is found that at $(M_0, 0, E_0)$

$$\left.\begin{array}{l}
F_{MM} = x_{MM}(3T\,;M_0,\,0,E_0) = 0, \quad F_{ME} = x_{ME}(3T\,;M_0,0,E_0) = 0\\
F_{EE} = x_{EE}(3T\,;M_0,0,E_0) = 0, \quad F_{NN} = x_{NN}(3T\,;M_0,0,E_0) = 0\\
F_{MN} = x_{MN}(3T\,;M_0,0,E_0) = -\dfrac{2(\ell^2-k^2)}{\dot{x}^0(t_1)}\left\{\dfrac{9}{4}\psi^2(t_1) - \psi^2(T)\varphi^2(t_1)\right\}\varphi(t_1),\\
F_{NE} = x_{NE}(3T\,;M_0,0,E_0)\\
\quad = -\dfrac{2(\ell^2-k^2)}{\dot{x}^0(t_1)}\left[\left\{2\psi^2(T)\varphi^2(t_1) + \dfrac{3}{2}\psi^2(t_1)\right\}x_E(t_1) + x_E(T)\varphi(t_1)\right.\\
\qquad \left.\left\{2\,\psi^2(T)\varphi^2(t_1) - \dfrac{1}{2}\,\psi^2(t_1)\right\}\right],
\end{array}\right\} \tag{22}$$

$$\left.\begin{array}{l}
G_{MM} = \dot{x}_{MM}(3T\,;M_0,0,E_0) = \dfrac{2(\ell^2-k^2)}{\dot{x}^0(t_1)}\cdot\dfrac{3}{4}\varphi(t_1)\left\{\varphi^2(t_1) - 4\,\dot{\varphi}^2(T)\,\psi^2(t_1)\right\},\\
G_{ME} = \dot{x}_{ME}(3T\,;M_0,0,E_0)\\
\quad = \dfrac{2(\ell^2-k^2)}{\dot{x}^0(t_1)}\left[\left\{\dfrac{3}{2}\varphi^2(t_1) + 2\dot{\varphi}^2(T)\,\psi^2(t_1)\right\}x_E(t_1) + x_E(T)\varphi(t_1)\right.\\
\qquad \left.\left\{\dfrac{1}{2}\,\varphi^2(t_1) + \dfrac{10}{3}\dot{\varphi}^2(T)\,\psi^2(t_1)\right\}\right],\\
G_{EE} = \dot{x}_{EE}(3T\,;M_0,0,E_0)\\
\quad = \dfrac{-2(\ell^2-k^2)}{\dot{x}^0(t_1)}\left[x_E(T)\left\{2\,\varphi^2(t_1) + \dfrac{8}{3}\dot{\varphi}^2(T)\,\psi^2(t_1)\right\}x_E(t_1) + x_E^2(T)\varphi(t_1)\right.\\
\qquad \left.\left\{\varphi^2(t_1) + \dfrac{28}{9}\,\dot{\varphi}^2(T)\,\psi^2(t_1)\right\}\right],\\
G_{NN} = \dot{x}_{NN}(3T\,;M_0,0,E_0) = \dfrac{2(\ell^2-k^2)}{\dot{x}^0(t_1)}\,\varphi(t_1)\left\{\dfrac{9}{4}\psi^2(t_1) - \psi^2(T)\varphi^2(t_1)\right\},\\
G_{MN} = \dot{x}_{MN}(3T\,;M_0,0,E_0) = 0, \quad G_{NE} = \dot{x}_{NE}(3T\,;M_0,0,E_0) = 0
\end{array}\right\} \tag{23}$$

Therefore the locus of $F = G = 0$ near $(M_0,\,0,\,E_0)$ consists of two branches, one tangent to the line $N = 0$, $(M-M_0) - \frac{2}{3} x_E(T)\,(E-E_0) = 0$ and the other tangent to the line $N=0, A\,(M-M_0) + B\,(E-E_0) = 0$, where

$$A = \frac{3}{4}\left\{\varphi^2(t_1) - 4\,\dot{\varphi}^2(T)\,\varphi^2(t_1)\right\}\varphi(t_1),$$

$$B = (3\,\varphi^2(t_1) + 4\,\dot{\varphi}^2(T)\,\psi^2(t_1))\,x_E(t_1) + x_E(T)\,\varphi(t_1)\,(\frac{3}{2}\varphi^2(t_1) + \frac{14}{3}\,\dot{\varphi}^2(T)\,\psi^2(t_1))$$

The first branch tangent to $N = 0, M - M_0 = \frac{2}{3}\,x_E(T)\,(E - E_0)$ is, of course, the curve of T-periodic solutions type ${}_1A$, the second branch tangent to the line $N = 0, A\,(M - M_0) + B\,(E - E_0) = 0$ is the trajectories of $\frac{1}{3}$-harmonics of type ${}_3A$.

Finally, we have the same results from numerical calculations shown in Fig. 5 (a), (b), (c), and (d) as predicted from qualitative discussion above.

5. Conclusions

In this paper the branching conditions are dealt with in regard to the bifurcation of $\frac{1}{3}$-harmonics in the neighborhood of branching point. And we have the results that the locus of $F = G = 0$ near bifurcation point consists of two branches, one corresponding to harmonic solution, the other $\frac{1}{3}$-harmonics.

Finally, it is noted that numerical calculations were performed by using ACOS-600 at the computer center, University of Osaka Prefecture.

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