Hessian Matrix for the Networks Containing Periodically operated Switches

Hessian Matrix for the Networks Containing Periodically Operated Switches

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In this paper, we will show that the Hessian matrix for the networks containing periodically operated ideal switches can be precisely calculated without any analytic expression, but in far less effort than that would be required by the pertubation method.

1. Introduction

For the optimization of electrical networks the gradient-based methods are normally used, because several efficient methods for calculating the gradients of a network function to a set of design parameters have been proposed. Among them the most widely used is that of Fletcher and Powell[1]. This requires only first derivatives (the gradients), but implicitly estimate second derivative information to speed convergence. Wing and Behar[2] showed that Hessian matrix (matrix of second derivatives of network function) can be obtained from first- and second-network sensitivities and can be used to solve optimization problems. David Agnew[3] developed an algorithm which can reduce the amount of work required to generate the Hessian. S.W. Director[1] showed the calculation procedure of second order sensitivities (the Hessian) with the aid of the adjoint network method, which is very useful for CAD.

However, we can't apply these methods listed above directly to the networks containing switches, for example, switched modulator, variable active switched filter etc..

This paper deals with the calculating procedure of the Hessian matrix in the frequency domain for the networks containing periodically operated *ideal* switches (switching networks). In a previous work[4], the authors derived the adjoint network representation for the linear netowrks containing switches. By the use of properties that exist between the switching network and its adjoint network, we obtain new formulas to determine the factors of Hessian matrix. The purpose of this paper is, first, to give the calculation algorithm of Hessian matrix; next, to estimate the total number of network analyses. And finally, a concrete example is shown to serve for an illustration.

2. Calculation of the gradients

In 4), we proposed the following definitions.

[*Definition* 1] *the adjoint network*

(1) The adjoint network $\hat{N}$ has the same topology as the original network N, and $\hat{N}$ is obtained from N by the branch replacement rules proposed by S.W. Director and

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R.A. Rohrer[5], hereat, each switch in N is also duplicated in $\hat{N}$.

(2) *ON*-switch in N corresponds to *ON*-switch in $\hat{N}$ and *OFF*-switch in N to *OFF*-switch in $\hat{N}$.

(3) It is forbidden to observe N and $\hat{N}$ at the same time when the positions of the corresponding switches in N and $\hat{N}$ are different.

[*Definition* 2] *the gradient*

The gradients in the switching network N are defined as the partial derivatives of the equivalent transfer function of N with respect to network parameters.

The outline of the gradient generations based on the adjoint network method will be described below.

Consider the time interval $nT_s \leqq t \leqq (n+1)T_s$ as shown in Fig. 1, where T_s is the

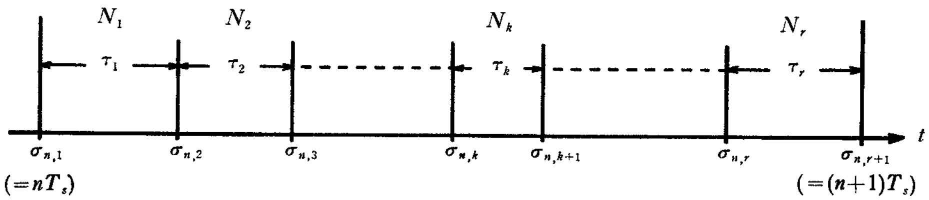

Fig. 1. Notations for n-th switching period.

switching period. Each switching period T_s is divided into r subintervals $\tau_1, \tau_2, \cdots, \tau_k, \cdots, \tau_r$, which represent r different sets of switch positions. For the k-th subinterval, the time invariant network N_k, where *ON*-switches are short-circuited and *OFF*-switches are open-circuited, is provided. As a result, the time invariant networks $N_1, N_2, \cdots, N_k, \cdots, N_r$ are provided in order of time according to the sets of switch positions. For the adjoint network, $\hat{N}_1, \hat{N}_2, \cdots, \hat{N}_k, \cdots, \hat{N}_r$ are also provided and they have one to one correspondence to $N_1, N_2, \cdots, N_k, \cdots, N_r$.

Consider a single input, single output linear network. By a suitable choice of excitations for N and $\hat{N}$, two analyses, one performed on N and another on $\hat{N}$, are sufficient

Table 1. Summary of $\partial(V_0, I_0)/\partial x$ calculation.

Element type x	Description in N and $\hat{N}$	$\partial(V_0, I_0)/\partial x$
impedance Z	$V_{Z,k}=ZI_{Z,k}$ $\hat{V}_{Z,k}=Z\hat{I}_{Z,k}$	$-\sum_{k=1}^{r}\frac{I_{Z,k}\hat{I}_{Z,k}}{\delta_k}$
admittance Y	$I_{Y,k}=YV_{Y,k}$ $\hat{I}_{Y,k}=Y\hat{V}_{Y,k}$	$\sum_{k=1}^{r}\frac{V_{Y,k}\hat{V}_{Y,k}}{\delta_k}$
inductance L	$V_{L,k}=j\omega_0 LI_{L,k}$ $\hat{V}_{L,k}=j\omega_0 L\hat{I}_{L,k}$	$-j\omega_0\sum_{k=1}^{r}\frac{I_{L,k}\hat{I}_{L,k}}{\delta_k}$
capacitance C	$I_{C,k}=j\omega_0 CV_{C,k}$ $\hat{I}_{C,k}=j\omega_0 C\hat{V}_{C,k}$	$j\omega_0\sum_{k=1}^{r}\frac{V_{C,k}\hat{V}_{C,k}}{\delta_k}$

to determine the gradients.

Here, we make an appointment that capital letters I and V indicate phasors. Let $(I_{x,k}, V_{x,k})$ and $(\hat{I}_{x,k}, \hat{V}_{x,k})$ be (current, voltage) of corresponding branch x in N_k and $\hat{N}_k$. Denote the output current and voltage by I_0 and V_0, respectively, the gradients $\partial(V_0, I_0)/\partial x$ with respect to various elements xs can be calculated by using the results in Table 1. In Table 1, $\delta_k = \tau_k/T_s$ $(k=1, 2, \cdots, r)$ and ω_0 is the angular frequency of input signal.

3. Algorithm to determine the Hessian matrix

We can observe that with network parameters $\boldsymbol{P}=[p_1, p_2, \cdots, p_i, p_j, \cdots, p_m]^T$ the Hessian matrix is of the form

$$\boldsymbol{H} = \begin{pmatrix} \dfrac{\partial^2(V_0, I_0)}{\partial p_1^2} & \dfrac{\partial^2(V_0, I_0)}{\partial p_2 \partial p_1} & \cdots\cdots & \dfrac{\partial^2(V_0, I_0)}{\partial p_m \partial p_1} \\ \dfrac{\partial^2(V_0, I_0)}{\partial_1 p \partial p_2} & \dfrac{\partial^2(V_0, I_0)}{\partial p_2^2} & \cdots\cdots & \vdots \\ \dfrac{\partial^2(V_0, I_0)}{\partial p_1 \partial p_m} & \cdots\cdots & \cdots\cdots & \dfrac{\partial^2(V_0, I_0)}{\partial p_m^2} \end{pmatrix} \qquad (1)$$

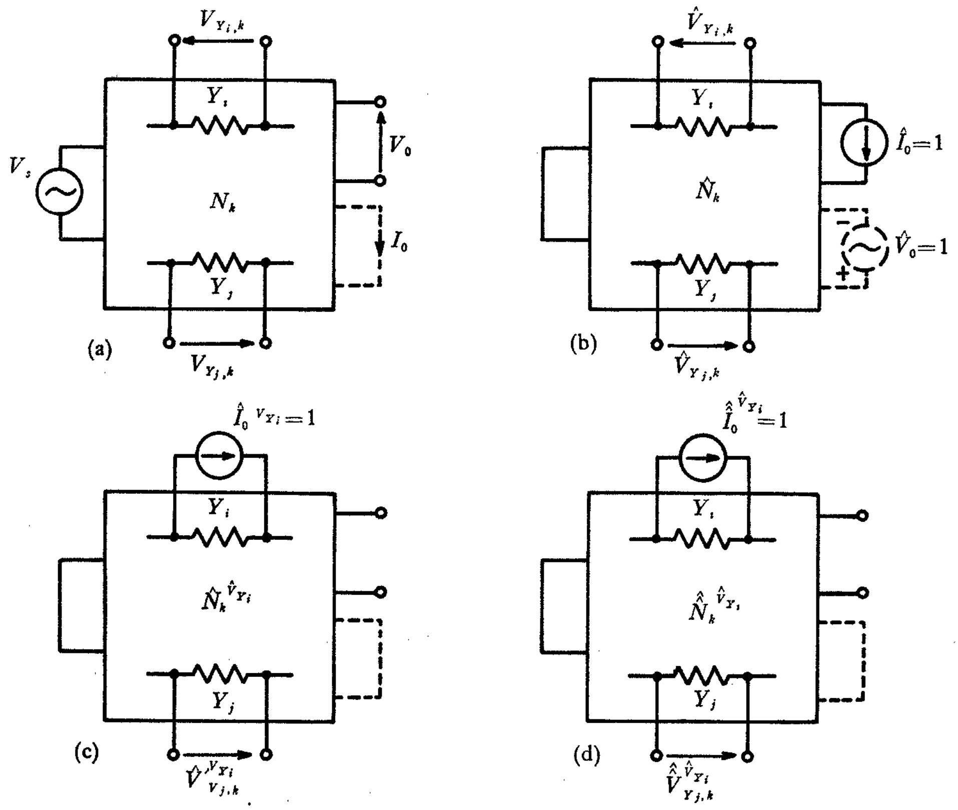

Fig. 2. Calculation of $\partial^2(V_0, I_0)/\partial Y_i \partial Y_j$.

In this section, we propose the calculation procedure of $\partial^2(V_0, I_0)/\partial p_i \partial p_j$. The model we propose for development of our calculation procedure is shown in Fig. 2(a), namely, a linear network N containing Y_i, Y_j, R, L, C, switches and various controlled sources etc., where $\partial^2(V_0, I_0)/\partial Y_i \partial Y_j$ or $\partial^2(V_0, I_0)/\partial Y_j \partial Y_i$ are to be found. In Fig. 2(a), V_s is the input voltage, $V_{Y_i,k}$ is the branch voltage across the admittance Y_i and $V_{Y_j,k}$ is that of Y_j. Fig. 2(b) shows the adjoint network. The unit current source $\hat{I}_0$ is added to $\hat{N}_k$ for the calculation of $\partial^2 V_0/\partial Y_i \partial Y_j$, while the unit voltage source $\hat{V}_0$ is applied to $\hat{N}_k$ for evaluating $\partial^2 I_0/\partial Y_i \partial Y_j$.

From Table 1, we write the gradient in the form

$$\frac{\partial(V_0, I_0)}{\partial Y_i} = \sum_{k=1}^{r} \frac{V_{Y_i,k}\hat{V}_{Y_i,k}}{\delta_k} \tag{2}$$

Then,

$$\frac{\partial}{\partial Y_j}\left(\frac{\partial(V_0, I_0)}{\partial Y_i}\right) = \sum_{k=1}^{r} \frac{(\partial V_{Y_i,k}/\partial Y_j)\hat{V}_{Y_i,k} + V_{Y_i,k}(\partial \hat{V}_{Y_i,k}/\partial Y_j)}{\delta_k} \tag{3}$$

Now $\partial V_{Y_i,k}/\partial Y_j$ and $\partial \hat{V}_{Y_i,k}/\partial Y_j$ can be calculated in the same manner as the product of two voltages. Fig. 2(c) is the adjoint network $\hat{N}_k^{V_{Y_i}}$ with $V_{Y_i,k}$ regarded as an "output" voltage.

Then,

$$\frac{\partial V_{Y_i,k}}{\partial Y_j} = \frac{V_{Y_j,k}\hat{V}^{V_{Y_i}}_{Y_j,k}}{\delta_k} \tag{4}$$

$$(k = 1, 2, \cdots, r)$$

where $\hat{V}^{V_{Y_i}}_{Y_j,k}$ is measured across Y_j-branch of $\hat{N}_k^{V_{Y_i}}$ but with the excitation $\hat{I}_0^{V_{Y_i}}=1$ in parall with Y_i (the output branch) as shown in Fig. 2(c). The term $\partial \hat{V}_{Y_i,k}/\partial Y_j$ can be similarly developed as

$$\frac{\partial \hat{V}_{Y_i,k}}{\partial Y_j} = \frac{\hat{V}_{Y_j,k}\hat{\hat{V}}^{\hat{V}_{Y_i}}_{Y_j,k}}{\delta_k} \tag{5}$$

$$(k = 1, 2, \cdots, r)$$

where $\hat{\hat{V}}^{\hat{V}_{Y_i}}_{Y_j,k}$ is measured across Y_j-branch of $\hat{\hat{N}}^{\hat{V}_{Y_i}}$, which is the adjoint network of $\hat{N}_k$ with the excitation $\hat{\hat{I}}_0^{\hat{V}_{Y_i}}=1$ in parallel with Y_i (the output branch), as shown in Fig. 2(d).

Substituting Eqs. (4) and (5) into Eq. (3)

$$\frac{\partial^2(V_0, I_0)}{\partial Y_j \partial Y_i} = \sum_{k=1}^{r} \frac{V_{Y_j,k}\hat{V}^{V_{Y_i}}_{Y_j,k}\hat{V}_{Y_i,k} + V_{Y_i,k}\hat{V}_{Y_j,k}\hat{\hat{V}}^{\hat{V}_{Y_i}}_{Y_j,k}}{\delta_k^2} \tag{6}$$

Similarly, we can obtain the next equation.

$$\frac{\partial^2(V_0, I_0)}{\partial Y_i \partial Y_j} = \sum_{k=1}^{r} \frac{V_{Y_i,k}\hat{V}^{V_{Y_j}}_{Y_i,k}\hat{V}_{Y_j,k} + V_{Y_j,k}\hat{V}_{Y_i,k}\hat{\hat{V}}^{\hat{V}_{Y_j}}_{Y_i,k}}{\delta_k^2} \tag{7}$$

From the properties of the adjoint network, we can obtain the following equations (Appendix 1)

$$\hat{V}^{V_{Y_j}}_{Y_i,k} = \hat{\hat{V}}^{\hat{V}_{Y_i}}_{Y_j,k} \tag{8}$$

$$\hat{V}^{V_{Y_i}}_{Y_j,k} = \hat{\hat{V}}^{\hat{V}_{Y_j}}_{Y_i,k} \tag{9}$$

Then, Eqs. (6) and (7) become

$$\frac{\partial^2(V_0, I_0)}{\partial Y_j \partial Y_i}\left(=\frac{\partial^2(V_0, I_0)}{\partial Y_i \partial Y_j}\right) = \sum_{k=1}^{r} \frac{V_{Y_i,k}\hat{V}_{Y_j,k}\hat{V}^{V_{Y_j}}_{Y_i,k} + V_{Y_j,k}\hat{V}_{Y_i,k}\hat{V}^{V_{Y_i}}_{Y_j,k}}{\delta_k^2} \tag{10}$$

This formula is symmetric in i and j as indeed it should be and can be calculated as the product of three voltages.

The results corresponding to Z, Y, L and C branches are shown in Table 2. Other types of branches can be handled in a similar manner.

Table 2. Summary of $\partial^2(V_0,I_0)/\partial x_i \partial x_j$ calculation.

	Z_j	Y_j
Z_i	$\sum_{k=1}^{r}\frac{I_{Z_i,k}\hat{I}_{Z_j,k}\hat{I}^{I_{Z_j}}_{Z_i,k} + I_{Z_j,k}\hat{I}_{Z_i,k}\hat{I}^{I_{Z_i}}_{Z_j,k}}{\delta_k^2}$	$-\sum_{k=1}^{r}\frac{I_{Z_i,k}\hat{V}_{Y_j,k}\hat{I}^{V_{Y_j}}_{Z_i,k} + V_{Y_j,k}\hat{I}_{Z_i,k}\hat{V}^{I_{Z_i}}_{Y_j,k}}{\delta_k^2}$
Y_i		$\sum_{k=1}^{r}\frac{V_{Y_i,k}\hat{V}_{Y_j,k}\hat{V}^{V_{Y_j}}_{Y_i,k} + V_{Y_j,k}\hat{V}_{Y_i,k}\hat{V}^{V_{Y_i}}_{Y_j,k}}{\delta_k^2}$

	L_j	C_j
Z_i	$j\omega_0\sum_{k=1}^{r}\frac{I_{Z_i,k}\hat{I}_{L_j,k}\hat{I}^{I_{L_j}}_{Z_i,k}+I_{L_j,k}\hat{I}_{Z_i,k}\hat{I}^{I_{Z_i}}_{L_j,k}}{\delta_k^2}$	$-j\omega_0\sum_{k=1}^{r}\frac{I_{Z_i,k}\hat{V}_{C_j,k}\hat{I}^{V_{C_j}}_{Z_i,k}+V_{C_j,k}\hat{I}_{Z_i,k}\hat{V}^{I_{Z_i}}_{C_j,k}}{\delta_k^2}$
Y_i	$-j\omega_0\sum_{k=1}^{r}\frac{V_{Y_i,k}\hat{I}_{L_j,k}\hat{V}^{I_{L_j}}_{Y_i,k}+I_{L_j,k}\hat{V}_{Y_i,k}\hat{I}^{V_{Y_i}}_{L_j,k}}{\delta_k^2}$	$j\omega_0\sum_{k=1}^{r}\frac{V_{Y_i,k}\hat{V}_{C_j,k}\hat{V}^{V_{C_j}}_{Y_i,k}+V_{C_j,k}\hat{V}_{Y_i,k}\hat{V}^{V_{Y_i}}_{C_j,k}}{\delta_k^2}$
L_i	$-\omega_0^2\sum_{k=1}^{r}\frac{I_{L_i,k}\hat{I}_{L_j,k}\hat{I}^{I_{L_j}}_{L_i,k}+I_{L_j,k}\hat{I}_{L_i,k}\hat{I}^{I_{L_i}}_{L_j,k}}{\delta_k^2}$	$\omega_0^2\sum_{k=1}^{r}\frac{I_{L_i,k}\hat{V}_{C_j,k}\hat{I}^{V_{C_j}}_{L_i,k}+V_{C_j,k}\hat{I}_{L_i,k}\hat{V}^{I_{L_i}}_{C_j,k}}{\delta_k^2}$
C_i		$-\omega_0^2\sum_{k=1}^{r}\frac{V_{C_i,k}\hat{V}_{C_j,k}\hat{V}^{V_{C_j}}_{C_i,k}+V_{C_j,k}\hat{V}_{C_i,k}\hat{V}^{V_{C_i}}_{C_j,k}}{\delta_k^2}$

The procedure for determining $\boldsymbol{H}$ of Eq. (1) can now be summarized.

Step 1 Determine all branch variables $V_{p_i,k}$, $V_{p_j,k}$, $I_{p_i,k}$ and $I_{p_j,k}$ ($k=1, 2, \cdots, r$) of the original network;

Step 2 determine all branch variables $\hat{V}_{p_i,k}$, $\hat{V}_{p_j,k}$, $\hat{I}_{p_i,k}$ and $\hat{I}_{p_j,k}$ ($k=1, 2, \cdots, r$) of the adjoint network;

Step 3 for any j, determine all branch variables $\hat{V}^{(V,I)_{p_j}}_{p_i,k}$ and $\hat{I}^{(V,I)_{p_j}}_{p_i,k}$ ($i=1, 2, \cdots, m$) in the analysis of adjoint network with a unit source across p_j, for $j=1, 2, \cdots, m$.

To obtain numerical solutions, we must calculate the branch variables in the k-th subinterval ($k=1, 2, \cdots, r$) for the original network, its adjoint network and the networks of adjoint type constructed by regarding each branch as the output branch. M.L. Liou's state variable approach[6] is suitable for the properties required. The summary is as follows.

For N_k, the differential state equation and the output equation are obtained as

$$\begin{cases} \dot{\boldsymbol{x}}_k = \boldsymbol{A}_k\boldsymbol{x}_k+\boldsymbol{B}_k\boldsymbol{u} \\ \boldsymbol{y}_k = \boldsymbol{C}_k\boldsymbol{x}_k+\boldsymbol{D}_k\boldsymbol{u} \end{cases} \quad (k = 1, 2, \cdots, r) \tag{11}$$

where $\boldsymbol{x}_k$ is the state variable, $\boldsymbol{u}$ is the input-function vector, $\boldsymbol{y}_k$ is the output-function vector, and $\boldsymbol{A}_k$, $\boldsymbol{B}_k$, $\boldsymbol{C}_k$, $\boldsymbol{D}_k$ are constants.

Substituting $\boldsymbol{A}_k$, $\boldsymbol{B}_k$, $\boldsymbol{C}_k$, $\boldsymbol{D}_k$ and other network parameters into M.L. Liou's equation (Appendix 2),

$$\boldsymbol{T}_k(p) = [\boldsymbol{C}_k e^{-p\eta_k}\boldsymbol{R}_k(p)-\tau_k\{\boldsymbol{C}_k(\boldsymbol{A}_k-P\boldsymbol{I})^{-1}\boldsymbol{B}_k-\boldsymbol{D}_k\}]/T_s \tag{12}$$

we can obtain the numerical solutions of all branch voltages and currents for N_k ($k=1, 2, \cdots, r$).

Similarly, we can calculate all branch variables for $\hat{N}_k$, $\hat{N}_k^{(V,I)p_i}$ and $\hat{N}_k^{(V,I)p_j}$.

4. Computational cost

It is common to regard the number of elementary arithmetic operations alone as the measure of the computational effort. In this section, we examine the question.

The symmetry of $\boldsymbol{H}$ requires the calculation of only the upper (or lower) triangular matrix. The number of operations required in each step to solve the Hessian matrix is shown in Table 3. Therefore, a total of $r(m+2)$ network analyses are required.

Table 3. Operation count.

step	network	no. of operations
1	N_k	r
2	$\hat{N}_k$	r
3	$\hat{N}_k^{(V,I)p_i}$	rm
total		$r(m+2)$

On the other hand, if the partial derivatives are calculated by successive differencing of p_j (pertubation method), i.e.

$$\frac{\partial^2(V_0, I_0)}{\partial p_i \partial p_j} = \frac{\left.\frac{\partial(V_0, I_0)}{\partial p_j}\right|_{p_i=p_i^0+\Delta p_i} - \left.\frac{\partial(V_0, I_0)}{\partial p_j}\right|_{p_i=p_i^0}}{\Delta p_i} \tag{13}$$

p_i^0; initial value

Δp_i; small change of p_i

a total of $m(m+2)/2$ separate analyses are required after the first derivatives are estimated, and the computational effort can rise sharply with the number of parameters and becomes intolerable for a large network. Since $r \ll m$, this method is far better than the pertubation method, which is commonly used, at the point of view of computational cost.

5. Example

For the simple lowpass filter of Fig. 3(a), calculate $\partial^2 V_0/\partial G_1 \partial G_2$. Fig. 3(b) is the adjoint network, Fig. 3(c) is the network built by taking G_1 as the output branch and Fig. 3 (d) is the adjoint network which results on regarding G_2 as the output branch.

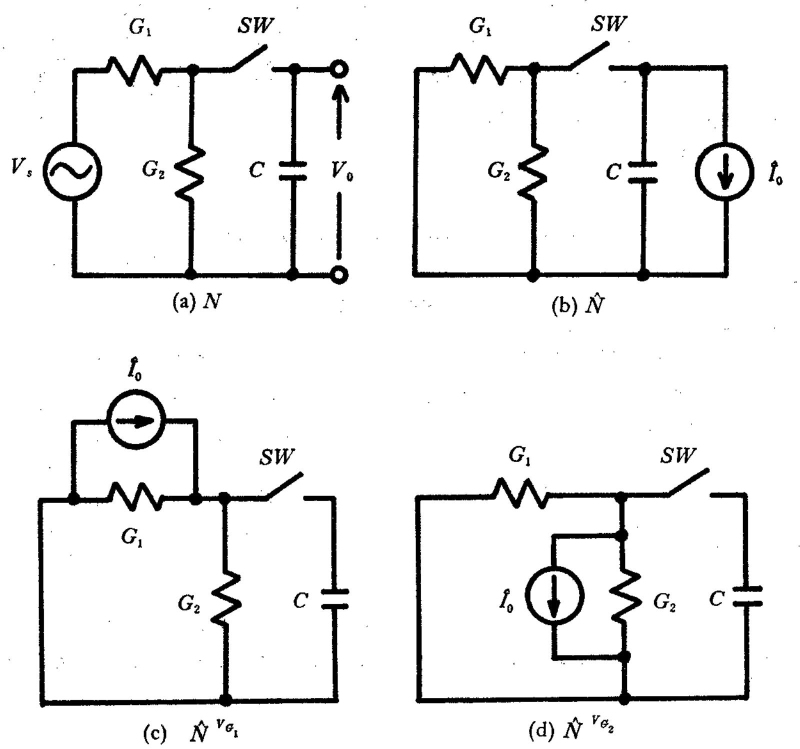

Fig. 3. Example.
$G_1 = G_2 = 1$ m℧, $C = 0.1$ μF
$\omega_0 = 2 \times 10^3$ rad/sec
$\omega = 2 \times 10^5$ rad/sec

Assume that the switch SW is *on* in the interval $[nT_s, nT_s+\tau_1)$ and *off* in the interval $[nT_s+\tau_1, (n+1)T_s)$, the original network N is divided into N_1 (SW is short-circuited) and N_2 (SW is open-circuited). Similarly, $\hat{N}$ is divided into $\hat{N}_1$ and $\hat{N}_2$, $\hat{N}^{V_{G_1}}$ into $\hat{N}_1^{V_{G_1}}$ and $\hat{N}_2^{V_{G_1}}$, $\hat{N}^{V_{G_2}}$ into $\hat{N}_1^{V_{G_2}}$ and $\hat{N}_2^{V_{G_2}}$.

Let the capacitor voltage be the state variable, we have the following equations by inspection

for N_1

$$\dot{v}_{C,1} = -\frac{G_1+G_2}{C} v_{C,1} - \frac{G_1}{C} v_s \tag{14}$$

$$v_{G_1,1} = -v_{C,1} + v_s \tag{15}$$

and for N_2

$$\dot{v}_{C,2} = 0 \tag{16}$$

$$v_{G_1,2} = \frac{G_2}{G_1+G_2} v_s \tag{17}$$

From these equations, the constants of Eq. (11) are determined. Substituting the constants $A_1=-(G_1+G_2)/C$, $B_1=-G_1/C$, $C_1=-1$, $D_1=1$, $A_2=0$, $B_2=0$, $C_2=0$, $D_2=G_1/(G_1+G_2)$ into Eq. (12), we can obtain the values of $V_{G_1,1}$ and $V_{G_1,2}$.

In the same manner, $V_{G_2,1}$ and $V_{G_2,2}$ can be calculated. For $\hat{N}$, $\hat{N}^{V_{G_1}}$ and $\hat{N}^{V_{G_2}}$ as well, we can obtain the numerical results of $\hat{V}_{G_1,k}$, $\hat{V}_{G_2,k}$, $\hat{V}^{V_{G_2}}_{G_1,k}$ and $\hat{V}^{V_{G_1}}_{G_2,k}$ $(k=1, 2)$ from their state equations and output equations. The results are shown in Table 4, where δ_1 is equal to τ_1/T_s.

Table 4. Numerical results of Example.

$E+n$; 10^n

δ	$\partial^2 V_0/\partial G_1 \partial G_2$	
	Calculation results	From Eq. (20)
1.0	$-0.62500E+7 + j0.62500E+7$	$-0.62500E+7 + j0.62500E+7$
0.8	$-0.33370E+7 + j0.68484E+7$	$-0.33371E+7 + j0.68484E+7$
0.6	$-0.28586E+6 + j0.56671E+7$	$-0.28562E+6 + j0.56674E+7$
0.4	$0.13327E+7 + j0.29107E+7$	$0.13326E+7 + j0.29111E+7$
0.2	$0.78220E+6 + j0.52611E+6$	$0.78232E+6 + j0.52629E+6$

This is a special case that the equivalent transfer function V_0 (suppose that the input V_s equals to 1) of this network can be written under the condition $\omega_0 \ll \omega$ $(\omega = 2\pi/T_s)$ as

$$V_0 = \frac{G_1}{p(C/\delta)+G_1+G_2} \tag{18}$$

where

$$p = j\omega_0 \tag{19}$$

The second order derivative of Eq. (19) with respect to G_1 and G_2 becomes

$$\frac{\partial^2 V_0}{\partial G_1 \partial G_2} = \frac{G_1-G_2-2p(C/\delta)}{\{p(C/\delta)+G_1+G_2\}^3} \tag{20}$$

Substituting the constants $G_1 = G_2 = 1\ m\mho$, $C=0.1\ \mu F$, $\omega_0 = 2\times 10^3\ rad/sec$ and $\omega = 2\times 10^5\ rad/sec$ into Eq. (20) the analytic results shown in Table 4 are obtained.

Comparing the results found by the adjoint network method with the analytic results, we can say that the present method does work with high accuracy.

6. Conclusions

The calculating procedure of the Hessian matrix for the networks containing periodically operated switches is presented. To calculate all factors of the Hessian matrix for the network with m parameters, a total of $r(m+2)$ networks must be analized, where r is the total number of the sets of switch positions. However, this method requires far less effort than would be required by the pertubation method for a large network.

This technique is applicable to any periodically switched network configurations and yields fast and accurate numerical results and, hence, is suitable for a large network optimizations.

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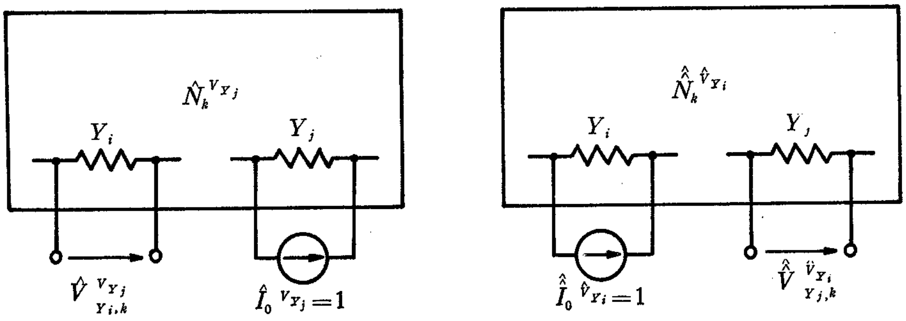
Appendix

Appendix 1

Fig. A(a) is the adjoint network by regarding the Y_j-branch voltage V_{Y_j} as the output voltage. Fig. A(b) is the adjoint network (of the adjoint network $\hat{N}_k$) by regarding the Y_i-branch voltage $\hat{V}_{Y_i}$ as the output voltage.



(a) (b)

Fig. A. Model to verify $\hat{V}^{V_{Y_j}}_{Y_i,k} = \hat{\hat{V}}^{\hat{V}_{Y_i}}_{Y_j,k}$.

The Tellegen sum for $\hat{N}^{V_{Y_j}}$ and $\hat{\hat{N}}^{\hat{V}_{Y_i}}$ is

$$I_0^{\hat{V}_{Y_i}} \cdot \hat{V}^{V_{Y_j}}_{Y_i,k} - V^{\hat{V}_{Y_i}}_{Y_j,k} \cdot \hat{I}_0^{V_{Y_j}} = 0 \qquad \text{(A-1)}$$

Since $\hat{I}_0^{V_{Y_j}}=1$ and $I_0^{V_{Y_i}}=1$, Eq. (A-1) becomes as

$$\hat{V}\,{}^{V_{Y_j}}_{Y_i,k} = \hat{\hat{V}}\,{}^{\hat{V}_{Y_i}}_{Y_j,k} \tag{A-2}$$

The same proof holds for

$$\hat{V}\,{}^{V_{Y_i}}_{Y_j,k} = \hat{\hat{V}}\,{}^{\hat{V}_{Y_j}}_{Y_i,k} \tag{A-3}$$

Appendix 2

For the k-th subinterval of n-th switching period shown in Fig. 1, the state equation and the output equation can be written as

$$\begin{cases} \dot{\boldsymbol{x}}_{n,k}(t) = \boldsymbol{A}_k\boldsymbol{x}_{n,k}(t)+\boldsymbol{B}_k\boldsymbol{u}(t) \\ \boldsymbol{y}_{n,k}(t) = \boldsymbol{C}_k\boldsymbol{x}_{n,k}(t)+\boldsymbol{D}_k\boldsymbol{u}(t) \end{cases} \tag{A-4}$$

$$\sigma_{n,k} \leqq t \leqq \sigma_{n,k+1}$$
$$(k = 1, 2, \cdots, r)$$

At the switching instants we write

$$\begin{cases} \boldsymbol{x}_{n,2}(\sigma_{n,2}) = \boldsymbol{F}_2\boldsymbol{x}_{n,1}(\sigma_{n,2})+\boldsymbol{G}_2\boldsymbol{u}(\sigma_{n,2}) \\ \boldsymbol{x}_{n,3}(\sigma_{n,3}) = \boldsymbol{F}_3\boldsymbol{x}_{n,2}(\sigma_{n,3})+\boldsymbol{G}_3\boldsymbol{u}(\sigma_{n,3}) \\ \qquad\vdots \\ \boldsymbol{x}_{n+1,1}(\sigma_{n,r+1}) = \boldsymbol{F}_{r+1}\boldsymbol{x}_{n,r}(\sigma_{n,r+1})+\boldsymbol{G}_{r+1}\boldsymbol{u}(\sigma_{n,r+1}) \end{cases} \tag{A-5}$$

where $\boldsymbol{F}_k$ and $\boldsymbol{G}_k$ are constants.

Let the input signal $u(t)$ be

$$u(t) = e^{pt}; \quad p = j\omega_0 \tag{A-6}$$

the equivalent transfer function $\boldsymbol{T}(p)$ becomes

$$\boldsymbol{T}(p) = \sum_{k=1}^{r} [\boldsymbol{C}_k e^{-p\eta_k}\boldsymbol{R}_k(p) - \tau_k\{\boldsymbol{C}_k(\boldsymbol{A}_k-p\boldsymbol{I})^{-1}\boldsymbol{B}_k-\boldsymbol{D}_k\}]/T_s \tag{A-7}$$

where

$$\eta_k = \sum_{i=1}^{k-1} \tau_i \tag{A-8}$$

$$\begin{aligned} \boldsymbol{R}_k(p) &= (\boldsymbol{A}_k-p\boldsymbol{I})^{-1}\{e^{A_k\tau_k}e^{-p\tau_k}-\boldsymbol{I}\}\boldsymbol{P}_k \\ &\quad +(\boldsymbol{A}_k-p\boldsymbol{I})^{-1}(\boldsymbol{A}_k-p\boldsymbol{I})^{-1}\{e^{A_k\tau_k}e^{-p\tau_k}-\boldsymbol{I}\}\boldsymbol{B}_k e^{p\eta_k} \end{aligned} \tag{A-9}$$

$$\boldsymbol{P}_1 = \boldsymbol{J}$$

$$\begin{aligned} \boldsymbol{P}_{k(\geq 2)} &= \boldsymbol{F}_k e^{A_{k-1}\tau_{k-1}}\boldsymbol{P}_{k-1}+\boldsymbol{F}_k(\boldsymbol{A}_{k-1}-p\boldsymbol{I})^{-1} \\ &\quad \times \{e^{A_{k-1}\tau_{k-1}}-e^{p\tau_{k-1}}\boldsymbol{I}\}e^{p\eta_{k-1}}\boldsymbol{B}_{k-1}+\boldsymbol{G}_k e^{p\eta_k} \end{aligned} \tag{A-10}$$

$$\boldsymbol{J} = (e^{pT_s}\boldsymbol{I}-\boldsymbol{M})^{-1}\boldsymbol{H} \tag{A-11}$$

$$\boldsymbol{M} = \boldsymbol{F}_{r+1}e^{A_r\tau_r}\boldsymbol{F}_r e^{A_{r-1}\tau_{r-1}}\cdots\boldsymbol{F}_2 e^{A_1\tau_1} \tag{A-12}$$

$$\boldsymbol{H} = \sum_{k=1}^{r} \boldsymbol{L}_k(\boldsymbol{A}_k-p\boldsymbol{I})^{-1}\{e^{A_k\tau_k}e^{p\eta_k}-e^{p\eta_{k+1}}\boldsymbol{I}\}\boldsymbol{B}_k+\sum_{k=1}^{r}\boldsymbol{N}_k e^{p\eta_{k+1}} \tag{A-13}$$

$$\begin{cases} \boldsymbol{L}_1 = \boldsymbol{F}_{r+1}e^{A_r\tau_r}\boldsymbol{F}_r e^{A_{r-1}\tau_{r-1}}\cdots\boldsymbol{F}_2 \\ \boldsymbol{L}_2 = \boldsymbol{F}_{r+1}e^{A_r\tau_r}\boldsymbol{F}_r e^{A_{r-1}\tau_{r-1}}\cdots\boldsymbol{F}_3 \\ \qquad\vdots \\ \boldsymbol{L}_r = \boldsymbol{F}_{r+1} \end{cases} \tag{A-14}$$

$$\left\{\begin{array}{l} \boldsymbol{N}_1 = \boldsymbol{F}_{r+1} e^{A_r \tau_r} \boldsymbol{F}_r e^{A_{r-1} \tau_{r-1}} \cdots \boldsymbol{F}_3 e^{A_2 \tau_2} \boldsymbol{G}_2 \\ \boldsymbol{N}_2 = \boldsymbol{F}_{r+1} e^{A_r \tau_r} \boldsymbol{F}_r e^{A_{r-1} \tau_{r-1}} \cdots \boldsymbol{F}_4 e^{A_3 \tau_3} \boldsymbol{G}_3 \\ \qquad \vdots \\ \boldsymbol{N}_{r-1} = \boldsymbol{F}_{r+1} e^{A_r \tau_r} \boldsymbol{G}_r \\ \boldsymbol{N}_r = \boldsymbol{G}_{r+1} \end{array}\right. \tag{A-15}$$

$\boldsymbol{I}$: unit (identity) matrix