Graph Theoretical Consideration on Interference between Simultaneous Two-Commodity Flows in a Directed Network

Graph Theoretical Consideration on Interference between Simultaneous Two-Commodity Flows in a Directed Network

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A conventional theorem is expanded by considering the interference between simultaneous two-cmmodity flows in a directed network from the graph theoretical point of view and it is definitely shown that a directed network, which the interference ratio of two-commodity flows is given by $m{:}n$ $(m, n = 1,2, ---)$, can be consituted.

The general algorithm to constitute a directed network which satisfies a specified domain of the existence of simultaneous two-commodity flows is also given.

1. Introduction

The multi-commodity flow problem in a communication network is one of the fundamental problems to be considered at all times in the analysis and design of the systems such as the flows of various kinds pass simultaneously through the same network.

In this paper, a directed network whose edges have given directions and capacities is dealt with, and the interference between two-commodity flows pass simultaneously through this network is considered from the graph theoretical point of view. The two-commodity flows come into the network at the different sources and run out from the different sinks of the network. It was already shown that such the simultaneous flows are possible to exist under some conditions[1]. As the result of detailed discussion to the same problem, we found by the graph theoretical consideration that it is necessary to rewrite a conventional theorem by a more generalized form. Needless to say, a new theorem in this paper contains the conventional one as a special case.

Furthermore, it is shown that a directed network which satisfies a specified domain of the existence of simultaneous two-commodity flows is possible to realize more generally.

2. Graph Theoretical Consideration

Let f_1 and f_2 be the values of simultaneous two-commodity flows F_1 and F_2. The conditions of the existence of these flows in a directed network is generally given by

$$f_1 \geqq 0, \qquad f_2 \geqq 0, \tag{1}$$

and

$$\alpha_k f_1 + \beta_k f_2 \leqq \gamma_k \quad (k = 1, 2, 3, \cdots), \tag{2}$$

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where α_k, β_k and γ_k are positive real numbers.

There is the following theorem representing the relation between α_k and β_k in Eq. (2):

[Theorem 1] The ratio of α_k to β_k is given by

$$\left.\begin{aligned} &\alpha_k : \beta_k = n : 1 \\ \text{or} \quad &\alpha_k : \beta_k = 1 : n \qquad (n = 0, 1, 2, \cdots). \end{aligned}\right\} \tag{3}$$

If we consider Eq. (3) from the graph theoretical point of view, it can be interpreted that since one directed path flowing F_1 (F_2) holds the edges in common with n directed paths flowing F_2 (F_1), the variation of f_1 (f_2) is magnified n times and interferes into f_2 (f_1).

In order to generalize such a consideration, we now consider the case where n directed paths flowing F_1 holds the edges in common with m directed paths flowing F_2. As a suitable example, the conditions of the existence of simultaneous two-commodity flows in a directed network as shown in Fig. 1 is discussed.

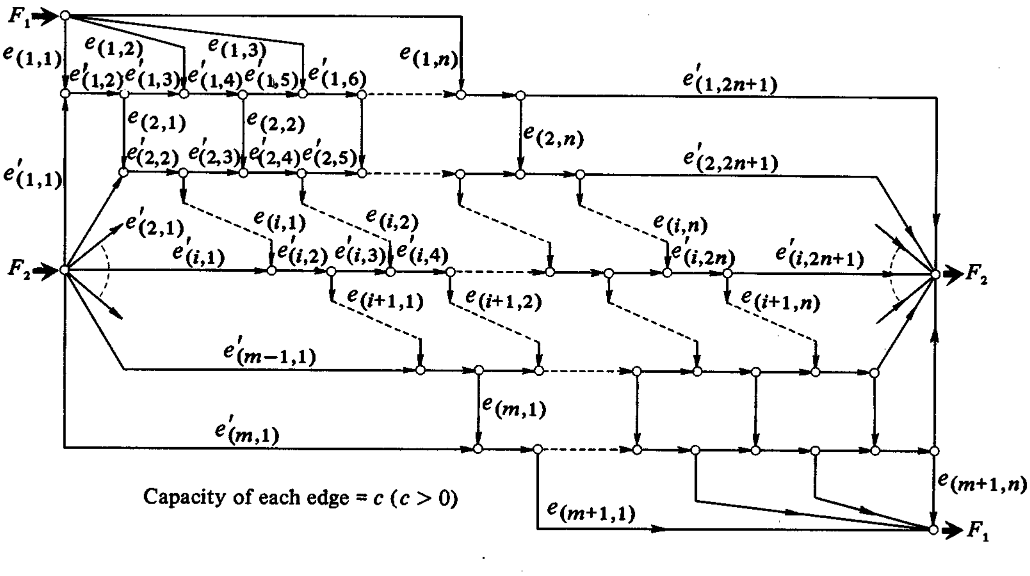

Fig. 1. Simultaneous two-commodity flows in a directed network.

First, the possible value of F_2 passing through the network in the case where the value of F_1 decreases by arbitrary value $n\epsilon$ ($0 \leq \epsilon \leq c$) from the maximum value of F_1 for $f_2 = 0$, nc, is determined. If and only if the value of flow passing through each of the directed paths $e_{(1,j)}$, $e'_{(1,2j)}$, $e_{(2,j)}$, $e'_{(2,2j)}$, $\cdots$, $e'_{(m,2j)}$ and $e_{(m+1,j)}$ ($j = 1, 2, \cdots, n$) is equal to the capacity c of each edge, the value of F_1 is the largest. Here we divide into the two following cases the method for making decrease the value of F_1 by $n\epsilon$:

(I) The case of making decrease of F_1 in each of n directed paths by ϵ, respectively;

In this case, it enables to pass ϵ at a time through each of the edges $e'_{(i,2)}$, $e'_{(i,4)}$, $\cdots$, $e'_{(i,2n-2)}$ and $e'_{(i,2n)}$ $(i = 1, 2, \cdots, m)$. Therefore, the possible value of F_2 passing through the network becomes $m\epsilon$ from the max-flow min-cut theorem[2)].

(II) The case of making decrease of F_1 by $n\epsilon$ by any other methods than that of (I);

This means that the value of F_1 in at least one edge among the edges $e_{(i,1)}$, $e_{(i,2)}$, $\cdots$, $e_{(i,n-1)}$ and $e_{(i,n)}$ is taken so as not to equal to $c-\epsilon$. In this case, the following lemmas hold;

[Lemma 1] Now let $e_{(i,j)}$ be the first edge, which the value of F_1 is over $c-\epsilon$, among the edges $e_{(i,1)}$, $e_{(i,2)}$, $\cdots$, $e_{(i,n-1)}$ and $e_{(i,n)}$, and also let $c-\epsilon_{(i,j)}$ $(0 \leqq \epsilon_{(i,j)} < \epsilon)$ be its value of F_1 in the edge $e_{(i,j)}$. Then the value of F_2 to be able to pass through the edge $e'_{(i,2j)}$ is at most $\epsilon_{(i,j)}$.

[Lemma 2] Under the same assumptions as those in Lemma 1, the average value of F_1 passing through each of the edges $e_{(i+1,j)}$, $e_{(i+1,j+1)}$, $\cdots$, $e_{(i+1,n-1)}$ and $e_{(i+1,n)}$ is larger than $c-\epsilon$.

[Lemma 3] The value of F_1 in at least one edge among the edges $e_{(i+1,j)}$, $e_{(i+1,j+1)}$, $\cdots$, $e_{(i+1,n-1)}$ and $e_{(i+1,n)}$ described in Lemma 2 is larger than $c-\epsilon$.

Using these lemmas, we can go through a procedure which is described in the following.

Let $e_{(1,j_1)}$ be the first edge, which the value of F_1 is over $c-\epsilon$, among the edges $e_{(1,1)}$, $e_{(1,2)}$, $\cdots$, $e_{(1,n-1)}$ and $e_{(1,n)}$, and also let $c-\epsilon_{(1,j_1)}$ be its value of F_1 in the edge $e_{(1,j_1)}$. Then the value of F_2 in the edge $e'_{(1,2j_1)}$ comes to at most $\epsilon_{(1,j_1)}$. If the values of F_1 in all edges are equal to $c-\epsilon$, the first edge $e_{(1,1)}$ is replaced by the edge $e_{(1,j_1)}$ and the value of $\epsilon_{(1,j_1)}$ is taken to ϵ. With similar way, the edge $e'_{(i,2j_i)}$ and the value of $\epsilon_{(i,j_i)}$ on the edges $e_{(i,j_{i-1})}$, $e_{(i,j_{i-1}+1)}$, $\cdots$, $e_{(i,n-1)}$ and $e_{(i,n)}$ $(i=2, 3, \cdots, m)$ are determined. Thus a directed cut-set with the edges $e'_{(1,2j_1)}$, $e'_{(2,2j_2)}$, $\cdots$, $e'_{(m,2j_m)}$ and $e_{(1,1)}$, $e_{(1,2)}$, $\cdots$, $e_{(1,j_1)}$, $e_{(2,j_1)}$, $\cdots$, $e_{(2,j_2)}$, $\cdots$, $e_{(m+1,j_m)}$, $e_{(m+1,j_m+1)}$, $\cdots$, $e_{(m+1,n)}$ is formed and the value of F_2 comes to $\sum_{i=1}^{m} \epsilon_{(i,j_i)}$. Since the value of $\epsilon_{(i,j_i)}$ is smaller than ϵ by the assumption, the following inequality holds:

$$\sum_{i=1}^{m} \epsilon_{(i,j_i)} < m\epsilon . \tag{4}$$

From the above consideration, we can understand that the case of (I) is the best method for maximizing F_2 when F_1 decreases by some value, that is, when

$$f_1 = nc - n\epsilon \qquad (0 \leqq \epsilon \leqq c), \tag{5}$$

$$f_2 \leqq m\epsilon . \tag{6}$$

From Eqs. (5) and (6), we have

$$mf_1 + nf_2 \leqq mnc . \tag{7}$$

Eqs. (1) and (7) represent the conditions of the existence of simultaneous two-commodity flows in the network of Fig. 1.

As the result of the above discussion, it can be seen that Theorem 1 should be rewritten as follows:

[Theorem 2] The relation between α_k and β_k in Eq. (2) is given by

$$\alpha_k : \beta_k = m : n , \tag{8}$$

where m and n are nonnegative integers.

3. Algorithm for Network Realization

A method for realization of the directed network which satisfied a specified domain of the existence of simultaneous two-commodity flows and its theoretical ground were already given[1),3)] under the conditions described in Theorem 1. By applying this method to the directed network of Fig. 1, we can describe it by a more generalized form.

We now give an algorithm introduced some modifications to the conventional one by considering saving of the edges.

[Algorithm]

Step 1. Obtain each of the cross points of the conditional equations, u^i (u_1^i, u_2^i) $(i = 1, 2, 3, \cdots)$, which given the domain of the existence of simultaneous two-commodity flows in the plane taking f_1 on the quardrature axis and f_2 on the longitude axis.

Step 2. Calculate each of the gradients of the conditional equations, $|u_2^{i+1} - u_2^i|$ $/ |u_1^{i+1} - u_1^i|$, and obtain their irreducible fractions m_i/n_i. In this case, if the denominator (numerator) is zero, the numerator (denominator) is set to 1.

Step 3. Constitute each of the network N_i corresponding to the conditional equations.

Step 4. Set to all $|u_2^{i+1} - u_2^i|/m_i$ the edge capacities of N_i. Take $|u_1^{i+1} - u_1^i|$ for $m_i = 0$ and $|u_2^{i+1} - u_2^i|$ for $n_i = 0$ as the edge capacities.

Step 5. Superimpose each of the network N_i.

4. Conclusion

In this paper, the interference between simultaneous two-commodity flows in a directed network has been discussed from the graph theoretical point of view and the conventional theorem has been expanded in a general form. The result described in this paper was adopted in other paper[4] which considered the feasibility condition of two-commodity flows in a directed network.

To find an algorithm for obtaining graph theoretically the condition of the existence of simultaneous flows in any directed network is the subject for a future study. In this case, the directed network of Fig. 1 will also be used in the process of cosideration and the result obtained in this paper seems to provide a very useful foundation.

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