An Alogebraic Approach to the Composition of Multi-valued Logical Functions

An Algebraic Approach to the Composition of Multi-valued Logical Functions

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This paper describes some algebraic properties of multi-valued logical functions and then shows an application of the properties to the minimal expression problem of logical functions. The relation of homomorphism of logical function is introduced. It is shown that the homomorphic relation is preserved by compositions of functions. Both a set of directed graphs which represent feed-forward logical circuits and an equivalence relation on the set are introduced. It is also shown that using the equivalence relation and the homomorphism, minimal expressions of multiple-valued logical functions are derivable from the minimal expressions of the simpler functions. Finally, an example is shown in which two minimal expressions of a two-valued logical function yield 12 minimal expressions of four-valued logical functions by the aid of the derived theorem.

1. Introduction

From an algebraic viewpoint, a logical function is considered as a mapping from a Cartesian product of a truth set M to M. From this point of view, the problems related to the composition of logical functions are important. The problems of the completeness of functions are studied by many researchers[1),2),3)]. On the other hand, the problems of compositions have not yet been studied so much[4),5)]. With regard to the relation between internal structures of functions and those of their compositions, few authors have investigated such a problem described in this paper. The authors introduce here the concept of homomorphism of logical functions and further, both a set of directed graphs which represent feed-forward logical circuits and an equivalence relation on the set. Using the equivalence relation and the homomorphism, the authors show that the minimal expressions of multiple-valued logical functions are derivable from the minimal expressions of the simpler functions.

2. Definitions of Fundamental Concepts

This chapter describes the definitions of logical functions and methods of expressing both logical functions and their compositions. Furthermore, some propositions derived from the definitions will also be shown.

Let M be a finite set and M^n be the n-th order Cartesian product of M. $\sharp(M)$ denotes the number of elements in M.

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[Definition 1] An n-variable logical function f on a set M is a mapping from M^n to M.

In this paper, two-variable functions are considered mainly, therefore the term "function" means a two-variable function. The function $f:M^2 \to M$ or $f(x, y)$ will be used to represent the function described above.

Let us write the pair of the set of functions F and the truth-set M on which the elements of F are defined, in parentheses as (F, M) and call it a function system.

To define the composition in a convenient form, we consider a type of acyclic directed graphs which satisfy the following three conditions:

(1) The graph C has $k+2$ ($k=1, 2, 3, \ldots$) nodes, where two of them have the incoming degree 0 and the others have the incoming degree 2.

(2) Among the nodes of incoming degree 2, there exists only one node of outgoing degree 0.

(3) The two arcs incoming to each node are distinguished as x-arc and y-arc corresponding to the representation $f(x, y)$.

Such a graph mentioned above is shown in Fig. 1.

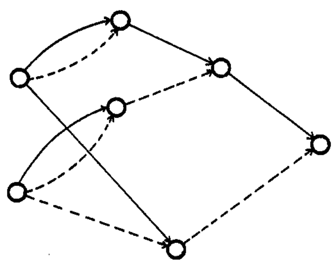

Fig. 1. An example of a directed graph.

In Fig. 1, $Z(C)$ denotes the number of nodes of C minus 2. Hereafter, the term "graph" will mean the acyclic directed graph mentioned above. If a function is assigned to each of the nodes of incoming degree 2, a feed-forward logical circuit is obtained. This circuit will realize a new function. Let us denote this function as $Cf(x, y)$ or Cf in short.

[Definition 2] A composition of a function $f(x, y)$ is $Cf(x, y)$ and the form of composition is C.

Let P^n be the set of graphs which satisfy $1 \leqq z(C) \leqq n$, and let S be a set such that $S=\{f' | f'=Cf, C \epsilon P^n\}$, then each of the two variable functions is considered as a mapping from P^n onto S.

[Definition 3] Let "$\underline{\underline{f}}$" denote a equivalence relation induced on the set P^n by $f:P^n \to S$, then the two graphs C_1 and C_2 in the relation $C_1 \underline{\underline{f}} C_3$ are called f-equivalent. In other words, $C_1 \underline{\underline{f}} C_2$ is $C_1 f = C_2 f$.

[Definition 4] $N(f^n)$ denotes the number of the two variable functions which are composed of k fs, where $1 \leqq k \leqq n$.

[Proposition 1] Let P^n/f denote the quotient set of P^n by the equivalence relation "$\underset{=}{f}$", then

$$N(f^n) = \sharp(P^n/f) \tag{1}$$

holds, where $\sharp(P^n/f)$ denotes the number of elements in the set P^n/f.

It is said that $f: M^2 \to M$ and $g: M'^2 \to M'$ are isomorphic, if $\sharp(M) = \sharp(M')$ holds and there exists a mapping T from M onto M' such that

$$T(f(x, y)) = g(T(x), T(y))$$

holds for any x and y in M. It is clear that if f and g are isomorphic to each other, then $N(f^n) = N(g^n)$ holds. To generalize the discussion, we newly introduce the idea of homomorphism of functions.

[Definition 5] Let f, g and T be $f: M^2 \to M$, $g: M'^2 \to M'$ and $T: M \to M'$ respectively, where T is an onto mapping. The function f is homomorphic to g if $T(f(x, y)) = g(T(x), T(y))$ holds for any x and y in M.

The relation $T(f(x, y)) = g(T(x), T(y))$ may be written as $Tf(x, y) = g(Tx, Ty)$ or $Tf = gT$ for simplicity.

[Definition 6] If a function $f: M^2 \to M$ is homomorphic to a function $g: M'^2 \to M'$ by a mapping T, then the system $(\{f\}, M)$ is said to be homomorphic to the system $(\{g\}, M')$ by T and relation is written as $(\{f\}, M) \underset{\sim}{T} (\{g\}, M')$. Furthermore, if T is a one-to-one mapping, then $(\{f\}, M)$ and $(\{g\}, M')$ are said to be isomorphic to each other and the relation is written as $(\{f\}, M) \underset{=}{T} (\{g\}, M')$

Let F_2^m be the set of all of the 2-variable m-valued functions.

[Definition 7] A two-variable m-valued function f is said to be complete if and only if its feed-forward compositions generate all of the elements of F_2^m.

[Proposition 2] A function $f \in F_2^m$ is complete if and only if there exists a natural number n such that $\sharp(P^n/f) = m^{m^2}$ holds.

[Definition 8] A function $f: M^2 \to M$ is said to be closed if there exists a nonempty subset D of M such that for any x and y in D, $f(x, y) \in D$ holds.

[Definition 9] A function $f: M^2 \to M$ is said to be decomposable, if there exists at least a decomposition $M(1)$, $M(2)$, ..., $M(n)$ such that $\bigcup_{i=1}^{n} M(i) = M$, $M(i) \cap M(j) = \phi$ and for a number i at least, $\sharp(M(i)) > 1$ holds, and furthermore if for every (i, j) $(1 \leqq i, j \leqq n)$ there exists $k (1 \leqq k \leqq n)$ such that for any elements $(x, y) \in (M(i), M(j))$, $f(x, y) \in M(k)$ holds.

3. Completeness and Homomorphism

Let us investigate some relations between completeness and homomorphism or isomorphism.

[Theorem 1] If $(\{f\}, M)\underline{T}(\{g\}, M')$ holds, then $TCf=CgT$ holds for every C in P^n.

(Proof) Suppose

$$T(f(x, y))=g(T(x), T(y)). \tag{2}$$

Furthermore, suppose there exist two function $F(x, y)$ and $G(x', y')$ such that

$$T(F(x, y))=G(T(x), T(y)) \tag{3}$$

holds for every (x, y) in M^2, then

$$T(F(x, y), y)=G(T(f(x, y)), T(y))=G(g(T(x), T(y)), T(y)), \tag{4}$$

$$T(F(x, f(x, y)))=G(T(x), T(f(x, y)))=G(T(x), g(T(x), T(y))) \tag{5}$$

and

$$T(F(f(x, y), f(x, y)))=G(T(f(x, y))) \\ =G(g(T(x), T(y)), g(T(x), T(y))) \tag{6}$$

hold. From Eqs. (2), (3), (4), (5) and (6), the theorem is evident by the mathematical inductions.

This theorem shows that the composition preserves the homomorphism. The next theorem shows a relation between the completeness and the homomorphism.

[Theorem 2] Suppose $f:M^2\rightarrow M$ is homomorphic to $g:M'^2\rightarrow M'$ and $\sharp(M)>\sharp(M')$ holds, then f is not complete.

(Proof) Since f becomes a decomposable function, it is not complete[2].

Since an isomorphism is a renaming of truth values, the next theorem is obtained directly.

[Theorem 3] If $(\{f\}, M)\underline{\underline{T}}(\{g\}, M')$ holds, then both f and g are either complete or incomplete.

The next theorem gives a sufficient condition of the incompleteness of a function.

[Theorem 4] If an additional condition $(\{f\}, M)\underline{\underline{U}}(\{g\}, M')$ holds in Theorem 3, then both f and g are incomplete.

(Proof) From the above assumption,

$$Tf(x, y)=g(Tx, Ty) \tag{7}$$

and

$$Uf(x, y)=g(Ux, Uy) \tag{8}$$

hold. Eq. (8) leads to

$$f(x, y)=U^{-1}g(Ux, Uy).$$

Substitution of this equation for $f(x, y)$ in Eq. (7) yields

$$TU^{-1}g(Ux, Uy)=g(Tx, Ty).$$

Respective replacements of Ux, Uy and TU^{-1} by x', y' and W result in

$$Wg(x', y')=g(Wx', Wy').$$

That is, g is isomorphic to itself, therefore g is incomplete. Then from the theorem 3,

f is also incomplete.

[Theorem 5] If $(\{f\}, M)\underline{\underline{T}}(\{g\}, M')$ and $(\{f\}, M)\underline{\underline{U}}(\{g\}, M')$ hold for both T and U, and if T is not equal to U, then f and g are isomorphic themselves by a pair of mapping which are conjugate to each other.

(Proof) In the proof of Theorem 4, deleting g from Eqs. (7) and (8) and replacing $U^{-1}T$ with V, we have

$$Vf(x, y)=f(Vx, Vy).$$

Furthermore, the equality relation

$$W=TU^{-1}=UU^{-1}TU^{-1}=UVU^{-1}=TU^{-1}TT^{-1}=TVT^{-1}$$

holds, that is, W and V are conjugate to each other.

The above result shows that a complete function on M has $\sharp(M)!$-1 isomorphic functions except for itself.

4. Some Fundamental Relations Between Compositions and Homomorphisms

It is known from Proposition 1 that the function $N(f^n)$ is decided by the character of the relation "$\underline{\underline{f}}$". This chapter clarifies some fundamental relations among "$\underline{\underline{f}}$", compositions and homomorphisms.

[Theorem 6] If $(\{f\}, M)\underline{\underline{T}}(\{g\}, M')$ holds, $C_1\underline{\underline{f}}C_2$ implies $C_1\underline{\underline{g}}C_2$ for any C_1 and C_2 in P^n.

(Proof) From the assumption and Theorem 1,

$$TC_1f(x, y)=C_2g(Tx, Ty)$$

and

$$TC_2f(x, y)=C_2g(Tx, Ty)$$

hold. The above relations yield

$$C_1g(Tx, Ty)=C_2g(Tx, Ty).$$

Since T is an onto mapping, the respective replacements Tx and Ty by x' and y' yield

$$C_1g(x', y')=C_2g(x', y').$$

That is, $C_1\underline{\underline{g}}C_2$ holds.

Theorem 6 shows that P^n/f is a refinement of P^n/g, that is, $N(f^n)=\sharp(P^n/f)\geqq \sharp(P^n/g)=N(g^n)$.

The converse of Theorem 6 does not hold generally.

[Theorem 7] If $(\{f\}, M)\underline{\underline{T}}(\{g\}, M')$ holds, then $P^n/f=P^n/g$ holds.

The converse of theorem 7 does not also hold, but if $\sharp(M)=\sharp(M')$ and f is complete, then there exists a one-to-one onto-mapping $T:M\rightarrow M'$ such that $(\{f\}, M)\underline{\underline{T}}(\{g\}, M')$ holds, as will be shown in Theorem 8.

The next lemma is evident from proposition 2.

[Lemma 1] Suppose $P^n/f=P^n/g$ holds for any n and $\sharp(M)=\sharp(M')$ holds, then if f is complete, g is also complete.

From the above lemma, we have the next lemma.

[Lemma 2] Suppose $\sharp(M)=\sharp(M')$ holds and $f:M^2\to M$ is complete and furthermore suppose $P^n/f=P^n/g$ holds for $g:M'^2\to M'$ and a sufficiently large n, then Cf is a constant if and only if Cg is a constant for any C in P^n.

(Proof) Let $[C]$ denote the graphs which are f-equivalent to C. Let M be a set $\{1, 2,\dots, k,\dots, m\}$. Suppose $C_kf=k$ for $C_k\epsilon[C_k]$ and $C_kg=h\neq$ *costant*. From Lemma 1. the completeness of g implies that a graph C_1 in P^n exists such that $C_1g \neq$ *constant* holds. Therefore, $C_kg(C_1g, C_1g)=h(C_1g, C_1g)\neq$ *constant* holds. That is, denoting the above composition as C_k', we have

$$C_k'\notin[C_k]. \tag{9}$$

Whereas, since $C_kf=k$, it follows that $C_kf(C_1f, C_1f)=k$ holds. Therefore

$$C_k'\epsilon[C_k] \tag{10}$$

holds. Eq. (9) contradicts Eq. (10).

[Theorem 8] Suppose that $f:M^2\to M$ and $g:M'^2\to M'$ are two functions on M and M', respectively, and that $\sharp(M)=\sharp(M')$ holds. If f is complete and $P^n/f=P^n/g$ holds for a sufficiently large n, then there exists exactly a one-to-one and onto mapping $T:M\to M'$ such that $(\{f\}, M)\ \underline{\underline{T}}(\{g\}, M')$ holds.

(Proof) Since the graphs which compose constant functions by the aid of f also compose constant functions by the aid of g from Lemma 2, the elements x_1, x_2 and x_3 in M, u_1, u_2 and u_3 in M' and C_1, C_2 and C_3 in P^n exist such that

$$x_1=C_1f, \qquad x_2=C_2f, \qquad x_3=C_3f \tag{11}$$

and

$$u_1=C_1g, \qquad u_2=C_2g, \qquad u_3=C_3g. \tag{12}$$

Furthermore, suppose

$$x_3=f(x_1, x_2) \tag{13}$$

and

$$u_1\neq g(u_1, u_2). \tag{14}$$

Then Eqs. (11), (12) and (13) yield

$$x_3=f(C_1f, C_2f). \tag{15}$$

That is

$$C_3'\epsilon[C_3], \tag{16}$$

where C_3' is the composition of Eq. (15).

On the other hand, Eqs. (11), (12) and (14) yield

$$u_3\neq g(C_1g, C_2g).$$

That is,

$$C_3'\notin[C_3]. \tag{17}$$

Eq. (16) contradicts Eq. (17). Therefore, the equality $x_3=f(x_1, x_2)$ corresponds to the equality $u_3=g(u_1, u_2)$. Since f is complete, f composes all of the constant functions, therefore, a one-to-one correspondence

$$T=\begin{pmatrix} x_1 & x_2 & x_3 \cdots\cdots x_m \\ u_1 & u_2 & u_3 \cdots\cdots u_m \end{pmatrix}$$

exists such that $f(x, y)=T^{-1}(g(T(x), T(y)).$

Therefore, $T(f(x, y))=g(T(x), T(y))$

holds. Furthermore, because of the completeness of f, T is unique according to Theorem 4.

[Theorem 9] Let $f{:}M^2\rightarrow M$ and $g{:}M'^2\rightarrow M'$ be functions which compose at least one constant function and suppose $\sharp(M)=\sharp(M')$ holds. Furthermore, suppose that both of the functions are not closed. Then $P^n/f=P^n/g$ for a sufficiently large n implies that f is isomorphic to g.

(Proof) Since f and g are not closed and compose at least one constant function, they compose all the constant functions. The latter part of proof is quite the same as that of theorem 8 except for the uniqueness of T.

In the theorem 9, let C_1f be a constant function, then using that f is not closed, $f(C_1f, C_1f)$ is not the same constant function as C_1f. Furthermore, $f(C_1f, f(C_1f, C_1f)), f(f(C_1f, C_1f), C_1f)$ or $f(f(C_1f, C_1f), f(C_1f, C_1f))$ contain at least one constatn function which is not equal to both $f(C_1f, C_1f)$ and C_1f. Thus, by writing n_1 for $z(C_1)$, all the constant functions are composed of the graphs with at most $2(2(\ldots(2(n+1)+1)\ldots+1)+1)+1+2=2^{m-1}n_1+2^{m-1}+1$ nodes. Therefore, the isomorphism is decided in finite steps.

From Theorem 9, the following corollary is evident:

[Corollary 1] Let f and g be two functions on M and M' respectively and let $\sharp(M)=\sharp(M')$ holds. Furthermore let each of them be not closed and compose at least one constant function, then the following two conditions are equivalent.

(i) $P^n/f=P^n/g$ holds for a sufficiently large n.

(ii) f is isomorphic to g.

A theorem which is analogous to Theorem 9 holds in the case of homomorphism. Before proving it, the following lemma is necessary:

[Lemma 3] Let $M=\{1, 2, \ldots, m\}$ and $M'=\{1, 2, \ldots, m'\}$ be two sets on which f and g are defined, respectively. Furthermore, suppose $m\geqq m'$. Then if $C_1 \underline{f} C_2$ implies $C_1 \underline{g} C_2$ for any C_1 and C_2 in P^n and if f and g compose all the constant functions, then

$$M \subset \bigcup_{i'=1}^{m'} [C_{i'}]_g f = \bigcup_{i'=1}^{m'} F_{i'} \qquad (18)$$

holds, where $[C_{i'}]_g$ denotes a set of graphs such that $C_{i'}g=i'$ and $i'\epsilon M'$ hold and $F_{i'}$ is the set $\{f'|f'=Cf,\ C\epsilon[C_i\,]_g\}$.

(Proof) Since $[C_{i'}]_g g=i'$ is a constant function, for two arbitrary elements $g'=[C']_g g$ and $g''=[C'']_g g$,

$$[C_{i'}]_g g(g', g'')=[C_{i'}]_g g$$

holds. Therefore

$$[C_{i'}]_g g([C']_g g, [C'']_g g)=[C_{i'}]_g g.$$

From the assumption that f is homomorphic to g, if $C_1 \underline{\underline{f}} C_2$ holds for any C_1 and C_2 in P^n, then $C_1 \underline{\underline{g}} C_2$ holds. Hence

$$[C_{i'}]_g f([C']_g f, [C'']_g f) \in [C_{i'}]_g f=F_{i'} \tag{19}$$

is derived. Since f composes all the constant functions, for any constant functions f_j and f_k in $S=\{f'|f'=Cf, C\in P^n\}$ Eq. (19) has to holds. Therefore, for any elements f_i in $F_{i'}$,

$$f_i(f_j, f_k) \in F_{i'} \tag{20}$$

holds. Since f_j and f_k are constant functions, $f_i(f_j, f_k)$ is also a constant function. Therefore, $F_{i'}$ contains at least one constant function. Thus, for each i' in M', $F_{i'}$ contains one constant function. Now, suppose that there exists a non-empty set L of constant functions in S such that

$$L \cap \bigcup_{i'=1}^{m'} F_{i'}=\phi. \tag{21}$$

Then there exists an element l in L such that $l(f', f'')=l$ for two arbitrary elements f' and f'' in S. Therefore, there exists a graph C_l in P^n such that $C_l f(f', f'')=C_l f$. That is, for two arbitrary elements C' and C'' in P^n,

$$C_l f(C'f, C''f)=[C_l]_f f.$$

Representing the above composition as C^*, we have

$$C^* \in [C_l]_f.$$

From the assumption that f is homomorphic to g, the above condition implies

$$C^* g=[C_l]_f g. \tag{22}$$

Eq. (22) shows that $[C_l]_f g$ is the compositions of a constant function. Therefore, we have $l \in F_l$, which contradicts Eq. (21).

Consequently, all the constant functions in S are contained in the set

$$\bigcup_{i'=1}^{m'} F_{i'}.$$

That is,

$$M \subset \bigcup_{i'=1}^{m'} F_{i'}.$$

[Theorem 10] Let $M=\{1, 2,\ldots, m\}$ and $M'=\{1, 2, \ldots, m\}$ be two sets such that $m \geqq m'$ holds and on which f and g are defined respectively, and suppose both of the functions compose all the constant functions. Then if for any C_1 and C_2

in P^n, $C_1 \underset{=}{f} C_2$ implies $C_1 \underset{=}{g} C_2$, then f is homomorphic to g.

(Proof) From lemma 3, each $F_{i'}(i'=1, 2, \ldots, m')$ satisfies

$$F_{i'} \cap F_{j'} = \phi$$

for $i' \neq j'$ in M'. Therefore by denoting each set of constant functions in the sets $F_{i'}, F_{j'}, \ldots, F_{k'}$ as $I_{i'}, I_{j'}, \ldots, I_{k'}$, the set M is decomposed as follows:

$$M = I_{i'} \cup I_{j'} \cup \ldots\ldots\ldots \cup I_{k'}. \qquad (23)$$

Furthermore, if $g(i', j') = k'$ holds, then

$$g([C_{i'}]_g g, [C_{j'}]_g g) = [C_{k'}]_g g \qquad (24)$$

holds. Therefore, we have

$$f(F_{i'}, F_{j'}) \in F_{k'},$$

which is rewritten as

$$f(I_{i'}, I_{j'}) \in I_{k'}. \qquad (25)$$

Eq. (25) shows that f is a decomposable function. Therefore let T be an mapping from M onto M' such that

$$T = \begin{pmatrix} I_1 & I_2 \ldots\ldots & I_{i'} \ldots\ldots & I_{m'} \\ 1 & 2 \ldots\ldots & i' \ldots\ldots & m' \end{pmatrix}$$

holds, then we have $T(f(x, y)) = g(T(x), T(y))$.

From the above theorem, the following corollary is evident:

[Corollary 2] Let $M = \{1, 2, \ldots, m\}$ and $M' = \{1, 2, \ldots, m'\}$ $(m \geqq m')$ be two sets on which two-variable functions f and g are defined respectively, then the following two conditions are equivalent.

(i) For any C_1 and C_1 in P^n, $C_1 \underset{=}{f} C_2$ implies $C_1 \underset{=}{g} C_2$.

(ii) f is homomorphic to g.

The above corollary shows a close relation between the internal structure of a function and the equivalence relation on the set of its compositions. The fact that "$\underset{=}{f}$ is a refinement of $\underset{=}{g}$" does not always implies "f is homomorphic to g" is evident from considerations about simple examples.

5. Homomorphism and Minimal Expressions

In this chapter, a relation between the homomorphism and the minimal expressions is investigated. A theorem concerning the minimal expressions is given and an example of its application is shown.

[Definition 10] Let f be a two-variable function defined on a set M and C be an element of P^n, then Cf is said to be a minimal expression if and only if for any C' in $[C]_f$, $z(C') \geqq z(C)$ holds.

From Corollary 2, the following theorem is obtained:

[Theorem 11] Let f and g be two-variable functions on the sets M and M', respectively. Suppose f is homomorphic to g, then for an arbitrary graph $C \in P^n$, if Cg is a minimal expression, then Cf is also a minimal expression.

(Proof) Let Cg be a minimal expression, then for an arbitrary graph $C' \in P^n$ which is g-equivalent to C, $z(C) \leqq z(C')$ holds. Suppose there exists a graph C'' which is f-equivalent to C and $Z(C'') < Z(C)$, then

$$Cf(x, y) = C''f(x, y) \tag{26}$$

holds. From the assumption, f is homomorphic to g, that is, $T:H \to M'$ exists such that

$$T(f(x, y)) = g(T(x), T(y)). \tag{27}$$

According to Corollary 2, we have the following equations from Eq. (27).

$$T(Cf(x, y)) = Cg(T(x), T(y)) \tag{28}$$

$$T(C''f(x, y)) = C''g(T(x), T(y)). \tag{29}$$

Eqs. (26), (28) and (29) lead the following equation,

$$Cg(T(x), T(y)) = C''g(T(x), T(y)).$$

Since T is an onto mapping, $T(x)$ and $T(y)$ are replaced by the two variables x' and y' in M', respectively. Therefore

$$Cg(x', y') = C''g(x', y')$$

holds. This shows that $C''g$ is g-equivalent to C and that $Z(C'') < Z(C)$, which contradict the assumption that Cf is a minimal expression.

[Example 2] Let f and g be such functions as given in Table 1.

Table 1. A function f which is homomorphic to g.

$M = \{1, 2, 3, 4\}$

f	1	2	3	4
1	4	4	2	1
2	4	3	1	1
3	2	2	2	2
4	2	1	2	1

$B = \{0, 1\}$

g	0	1
0	1	0
1	0	0

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Since $Tf = gT$ holds, the functions composed of f are the members of $[C]_g f$ according to Corollary 2. The minimal expressions by f are also in $[C]_g f$. The following two graphs C_1 and C_2 give the minimal expression of h, where h is given in Table 2.

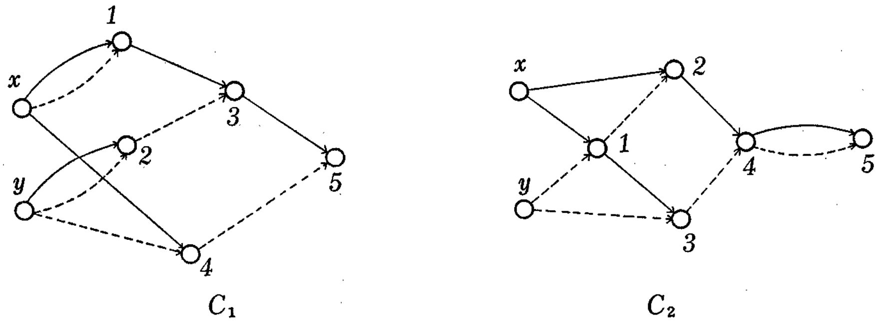

Fig. 2. Two graphs corresponding to minimal expressions.

Table 2. A function h which is obtained by g, C_1 and C_2.

h	0	1
0	0	1
1	1	0

Table 3. Twelve minimal expressions by f in the table 1.

x	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
y	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
C_1	1	1	4	4	1	1	4	4	4	4	2	1	4	4	1	2
$C_1(3)$	1	1	4	4	1	1	4	4	4	3	2	1	3	4	1	2
$C_1(5)$	2	1	4	4	1	2	4	4	4	4	1	1	4	4	1	1
$C_1(3, 5)$	2	1	3	4	1	2	4	4	4	3	1	1	3	4	1	1
C_2	1	1	3	3	1	1	4	3	4	4	1	1	4	4	1	1
$C_2(1)$	1	1	3	3	1	1	3	3	4	4	1	1	4	4	1	1
$C_2(2)$	2	1	3	3	1	2	4	3	4	3	1	1	4	4	1	1
$C_2(1, 2)$	2	1	3	3	1	2	3	3	4	4	1	1	4	4	1	1
$C_2(2, 3)$	1	1	4	4	1	1	4	4	4	3	1	1	4	4	1	1
$C_2(1, 2, 3)$	1	1	4	3	1	1	3	4	4	4	1	1	4	4	1	1
$C_2(3)$	1	1	4	4	1	1	4	4	4	4	2	1	4	4	1	2
$C_2(1, 3)$	1	1	4	3	1	1	3	4	4	4	2	1	4	4	1	2

Exchanging x-arc and y-arc of each node, we obtain 12 minimal expressions by f, as shown in Table 3.

In Table 3, $C(i, j, k, \ldots)$ denotes a graph which is obtained from a graph C by the exchanges of x-arc and y-arc on its nodes $i, j, k, \ldots$. These functions corresponding to the above graphs are different from each other and they are realized minimally.

6. Concluding Remarks

We have considered some algebraic properties of compositions of multiple-valued logical functions to obtain some basic relations between a function and compositions by the function. Theorem 10 or Corollary 2 is a homomorphism theorem of the multi-valued logical functions. In Chapter 5, an application of Theorem 11 which is derived from Corollary 2 is shown. It shows one of the approaches to the minimization problem of multi-valued logical circuits. By the aid of Theorem 11, complex minimization problems are reduced to simple ones if a condition is satisfied. Since theorem 11 is not sufficient to the general cases, the extension of it is a problem of future investigations. One of the approach is to extend the theorem to the case in which some don't-cares exist.

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