A Method for Calculating Multi-Dimensional Gaussian Distribution

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(Received November 15, 1975)

A method for calculating multi-dimensional Gaussian distributions is proposed, using the Hermite polynomial expansion method. An algorithmic procedure is developed for calculating probability distribution functions of arbitrary dimensions taking account of moment terms up to an arbitrary order. Numerical examples are provided to demonstrate the applicability of the proposed procedure.

1. Introduction

Multi-dimensional Gaussian distribution has been widely used in science[1] and engineering[2], and its properties are discussed in many literatures on statistics[3)4]. For the calculation of probability distribution functions (*p.d.f.*), multiple integrals must be performed by direct numerical integration or by transforming them into single integral using orthogonal transformation. However, it does not seem to the authors that these methods may be efficient for a fast numerical calculation of multi-dimensional Gaussian probability distribution functions.

In this paper, the Hermite polynomial expansion method is applied for calculating multi-dimensional Gaussian distributions. Through this method, multiple integrals encountered in the calculation of the *p.d.f.* are reduced to term-by-term integration of one variable, which saves greatly computational efforts. An algorithmic procedure is developed for calculating the *p.d.f.* of arbitrary dimensions taking account of the moment terms up to an arbitrary order. Numerical examples are presented for demonstrating the applicability of the proposed method. Are discussed the effects of correlation coefficients, the order of mement terms and the dimensions of the *p.d.f.* on the resulting values of the *p.d.f.* and the computer processing time.

2. Hermite Polynomial Expansion of Multi-dimensional Gaussian Distribution

The probability density function of a k-dimensional Gaussian distribution is written in the form

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$$P(x_1, x_2, \cdots, x_k) = \frac{1}{(2\pi)^{k/2}|\boldsymbol{V}|^{1/2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}\left(\frac{x_i-\mu_i}{\sigma_i}\right)\left(\frac{x_j-\mu_j}{\sigma_j}\right)\right]$$
$$= \frac{1}{(2\pi)^{k/2}|\boldsymbol{V}|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{V}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right], \quad (1)$$

where μ_i : mean of the random variable $X_i (i=1, 2, \cdots, k)$,
σ_i^2 : variance of $X_i (i=1, 2, \cdots, k)$,
$\boldsymbol{\mu}=\{\mu_i\}$: k-dimensional vector of the mean of X_i,
$\boldsymbol{x}=\{x_i\}$: k-dimensional vector of the realization of X_i,
$\boldsymbol{V}=[V_{ij}]$: $k \times k$ dimensional variance (V_{ii})-covariance (V_{ij}) matrix of X_i,
a_{ij}: coefficient determined by variance-covariance matrix,
$|[\ \ ]|$: determinant of a matrix [ ],
$\{\ \ \}^T$: transpose of a vector { },
$[\ \ ]^{-1}$: inverse of a matrix [ ].

The random variables X_i are standardized by the following transformation without losing generality:

$$Z_i = (X_i - \mu_i)/\sigma_i \qquad (i = 1, 2, \cdots, k). \quad (2)$$

Thus, $X_i (i=1, 2, \cdots, k)$ is used as standardized variables in place of Z_i in the following. The probability density function is written as

$$P(x_1, x_2, \cdots, x_k) = \frac{1}{(2\pi)^{k/2}|\boldsymbol{C}|^{1/2}} \exp\left[-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{C}^{-1}\boldsymbol{x}\right], \quad (3)$$

where $\boldsymbol{C}=[\rho_{ij}]$: $k \times k$ matrix of the correlation coefficients ρ_{ij} between X_i and X_j with $\rho_{ii}=1$ $(i, j=1, 2, \cdots, k)$.

The characteristic function corresponding to Eq. (3) is given by

$$\psi(t_1, t_2, \cdots, t_k) = \exp\left(-\frac{1}{2}\boldsymbol{t}^T\boldsymbol{C}\boldsymbol{t}\right)$$
$$= \exp\left(-\frac{1}{2}\boldsymbol{t}^T\boldsymbol{t}\right) \exp\left(-\sum_{i,j=1(j>i)}^{k} \rho_{ij}t_it_j\right), \quad (4)$$

where $\boldsymbol{t}=\{t_i\}$ is a k-dimensional vector of dummy variables $t_i (i=1, 2, \cdots, k)$. Expanding the second exponential function in Eq. (4) into a power series, the characteristic function is written as[4]

$$\psi(t_1, t_2, \cdots, t_k)$$
$$= \exp\left[-\frac{1}{2}\boldsymbol{t}^T\boldsymbol{t}\right] \times$$
$$\times \sum\left[(-1)^{m/2}\frac{\rho_{12}^{m_{12}}\rho_{13}^{m_{13}}\cdots\rho_{(k-1)k}^{m_{(k-1)k}}}{m_{12}!\,m_{13}!\cdots m_{(k-1)k}!}\right] t_1^{m_1}t_2^{m_2}\cdots t_k^{m_k}, \quad (5)$$

where the summation is over all possible sets of the $\rho_{ij} (i=1, 2, \cdots, k-1; j=i+1, i+2,$

$\cdots, k)$ taken over all non-negative values of the m_{ij}:

$$\left.\begin{aligned} m_i &= \sum_{j=1}^{i-1} m_{ji} + \sum_{j=i+1}^{k} m_{ij} , \\ m &= \sum_{i=1}^{k} m_i . \end{aligned}\right\} \tag{6}$$

Let N be defined by

$$N = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} m_{ij} , \tag{7}$$

m in Eq. (6) can be written as

$$\begin{aligned} m &= \sum_{i=1}^{k} \{\sum_{j=1}^{i-1} m_{ji} + \sum_{j=i+1}^{k} m_{ij}\} \\ &= \sum_{i=2}^{k} \sum_{j=1}^{i-1} m_{ji} + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} m_{ij} \\ &= 2 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} m_{ij} \\ &= 2N . \end{aligned} \tag{8}$$

where $m_{\cdot(k+1)} = m_{0\cdot} = 0$. Thus it is seen that m is zero or an even number. Using N defined by Eq. (7), the summation in Eq. (5) can be written in the form:

$$\begin{aligned} \psi(t_1, t_2, \cdots, t_k) &= \exp\left[-\frac{1}{2}\boldsymbol{t}^T\boldsymbol{t}\right] \times \\ &\times \sum_{N=0}^{\infty} \sum_{(m_{ij})_N} (-1)^N \frac{\rho_{12}^{m_{12}} \rho_{13}^{m_{13}} \cdots \rho_{(k-1)k}^{m_{(k-1)k}}}{m_{12}!\, m_{13}! \cdots m_{(k-1)k}!} t_1^{m_1} t_2^{m_2} \cdots t_k^{m_k} , \end{aligned} \tag{9}$$

where $\sum_{(m_{ij})_N}$ denotes a summation taken over all sets of non-negative values of the m_{ij} which satisfy the relation (7) for a given N.

Fourier inversion of $\psi(t_1, t_2, \cdots, t_k)$ gives the probability density function:

$$\begin{aligned} p(x_1, x_2, \cdots, x_k) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} \exp\left[-i\boldsymbol{t}^T\boldsymbol{x}\right] \times \\ &\times \psi(t_1, t_2, \cdots, t_k) dt_1 dt_2 \cdots dt_k \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} \exp\left[-i\boldsymbol{t}^T\boldsymbol{x} - \frac{1}{2}\boldsymbol{t}^T\boldsymbol{t}\right] \times \\ &\times \sum_{N=0}^{\infty} \sum_{(m_{ij})_N} (-1)^N \left[\frac{\rho_{12}^{m_{12}} \rho_{13}^{m_{13}} \cdots \rho_{(k-1)k}^{m_{(k-1)k}}}{m_{12}!\, m_{13}! \cdots m_{(k-1)k}!}\right] t_1^{m_1} t_2^{m_2} \cdots t_k^{m_k} dt_1 dt_2 \cdots dt_k . \end{aligned} \tag{10}$$

Interchanging summation and integration, Eq. (10) yields

$$\begin{aligned} p(x_1, x_2, \cdots, x_k) &= \sum_{N=0}^{\infty} \sum_{(m_{ij})_N} (-1)^N \left[\frac{\rho_{12}^{m_{12}} \rho_{13}^{m_{13}} \cdots \rho_{(k-1)k}^{m_{(k-1)k}}}{m_{12}!\, m_{13}! \cdots m_{(k-1)k}!}\right] \times \\ &\times \prod_{j=1}^{k} \int_{-\infty}^{\infty} \frac{1}{2\pi} t_j^{m_j} \exp\left[-it_j x_j - \frac{1}{2} t_j^2\right] dt_j . \end{aligned} \tag{11}$$

The integral in Eq. (11) is rewritten as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} t_j^{m_j} \exp\left[-it_jx_j-\frac{1}{2}t_j^2\right]dt_j$$
$$=\frac{(-i)^{m_j}}{2\pi}\int_{-\infty}^{\infty}\left(\frac{d}{dx_j}\right)^{m_j}\exp\left[-it_jx_j-\frac{1}{2}t_j^2\right]dt_j$$
$$=(-i)^{m_j}\left(\frac{d}{dx_j}\right)^{m_j}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[-it_jx_j-\frac{1}{2}t_j^2\right]dt_j$$
$$=(-i)^{m_j}\left(\frac{d}{dx_j}\right)^{m_j}\phi(x_j)\,, \tag{12}$$

where
$$\phi(x_j)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_j^2\right). \tag{13}$$

The derivatives of $\phi(x_j)$ are related to Hermite polynomials by

$$\left(\frac{d}{dx}\right)^n\phi(x)=(-1)^nH_n(x)\phi(x)\,. \tag{14}$$

Using Eqs. (6), (8), (11), (12) and (14), the probability density function $p(x_1, x_2, \cdots, x_k)$ is expressed as

$$p(x_1, x_2, \cdots, x_k)=\sum_{N=0}^{\infty}\sum_{(m_{ij})_N}\left[\frac{\rho_{12}^{m_{12}}\rho_{13}^{m_{13}}\cdots\rho_{(k-1)k}^{m_{(k-1)k}}}{m_{12}!\,m_{13}!\cdots m_{(k-1)k}!}\right]\prod_{j=1}^{k}H_{m_j}(x_j)\phi(x_j)\,. \tag{15}$$

Using Eqs. (14) and (15), the probability distribution function is given by

$$P(x_1, x_2, \cdots, x_k)=\sum_{N=0}^{\infty}\Delta P_{2N}\,, \tag{16}$$

$$\Delta P_{2N}=\sum_{(m_{ij})_N}\frac{\rho_{12}^{m_{12}}\rho_{13}^{m_{13}}\cdots\rho_{(k-1)k}^{m_{(k-1)k}}}{m_{12}!\,m_{13}!\cdots m_{(k-1)k}!}\prod_{j=1}^{k}(-1)H_{m_j-1}(x_j)\phi(x_j)\,, \tag{17}$$

where
$$(-1)H_{-1}(x_j)\phi(x_j)=\Phi(x_j)=\int_{-\infty}^{x_j}\phi(t)dt\,. \tag{18}$$

The above relations are easily programmed for digital computers, and a general computational algorithm is given in the following section.

3. Algorithmic Procedure

Using the relations (7), (16), (17) and (18), an algorithmic procedure can be developed for calculating the multi-dimensional Gaussian probability distribution functions taking account of the moment terms to any order. The procedure consists of the following steps.

Step 1. Specify the dimension (k), the order of the moment terms retained (NMT) and the value of x_i to calculate the $p.d.f.$.

Step 2. Set $P_0=\prod_{i=1}^{k}\Phi(x_i)$ and $N=0$.

Step 3. Set $N=N+1$ and perform the summation

$$P=\sum_{\{m_{ij}\}_N}\frac{\rho_{12}^{m_{12}}\rho_{13}^{m_{13}}\cdots\rho_{(k-1)k}^{m_{(k-1)k}}}{m_{12}!\,m_{13}!\cdots m_{(k-1)k}!}\prod_{j=1}^{k}(-1)H_{m_j-1}(x_j)\phi(x_j)$$

for all possible sets of non-negative values of the m_{ij} which satisfy Eq. (7) for the given N. Putting $P_{2N}=P_{2N-2}+P$, go to Step 4.

Step 4. If $N=NMT$, stop the calculation. Otherwise, go to Step 3.

The flow chart is given in Fig. 1, which illustrates the computational procedure

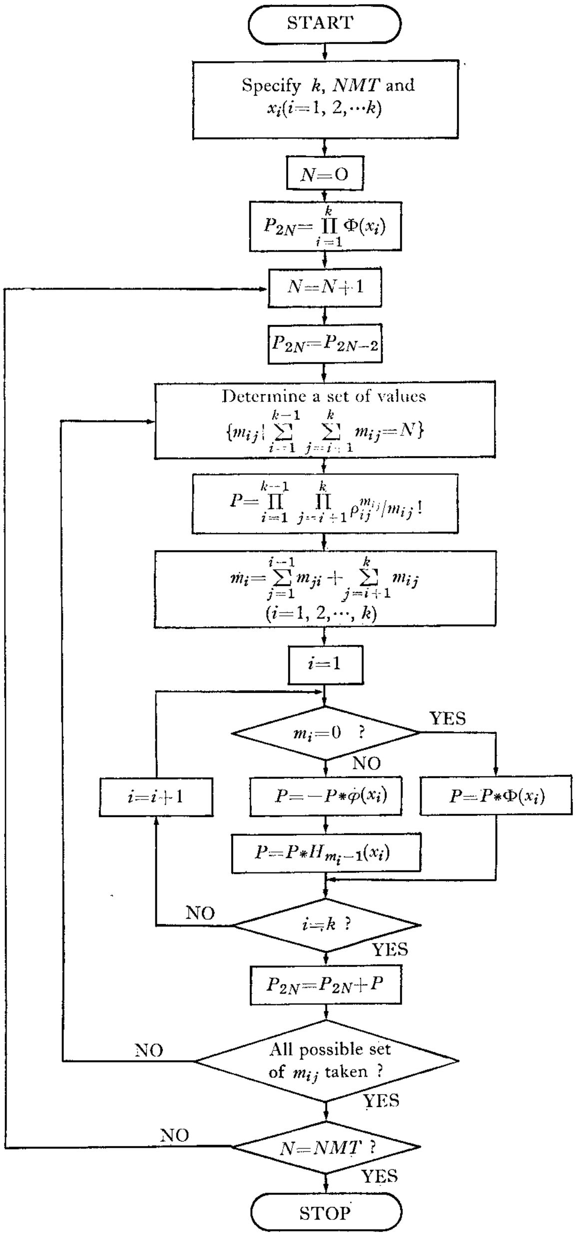

Fig. 1. Flow chart illustrating the computational procedure.

mentioned above.

4. Numerical Examples

First consider a two dimensional case. The values of the probability distribution function

$$P(x_1, x_2) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} P(x_1, x_2)\,dx_1\,dx_2 \tag{19}$$

are calculated for various values of the correlation coefficient ρ_{12}, retaining the moment terms up to the 40th-order ($N=20$). The results are plotted in Fig. 2. When ρ_{12} is positive, the values of $P(0, x_2)$ are greater than those in case of independent Gaussian

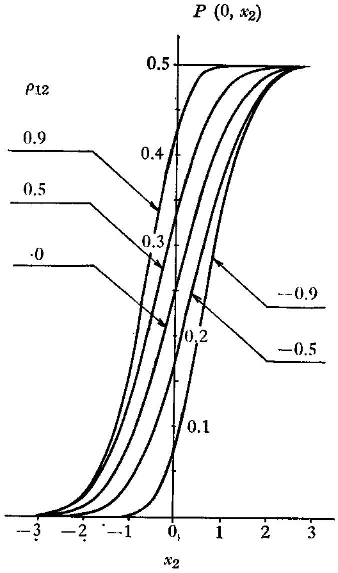

Fig. 2. Two-dimensional Gaussian *p.d.f.* for various values of correlation coefficient.

Table 1. Effect of the correlation coefficient on the resulting two-dimensional probability ($N=20$).

(a) $x_1=0$

ρ_{12} \ x_2	-3	-2	-1	0	1	2	3
0.7	0.0013487	0.0224614	0.1454782	0.3734070	0.4868229	0.4997113	0.4999988
0.5	0.0013089	0.0207236	0.1273982	0.3333333	0.4687429	0.4979735	0.4999591
0.3	0.0011450	0.0175397	0.1082745	0.2984933	0.4496193	0.4947896	0.4997951
0.1	0.0008494	0.0135181	0.0889808	0.2659421	0.4303256	0.4907681	0.4994995
0.	0.0006749	0.0113750	0.0793276	0.2500000	0.4206724	0.4886249	0.4993251
-0.1	0.0005004	0.0092319	0.0696744	0.2340579	0.4110192	0.4864818	0.4991506
-0.3	0.0002048	0.0052104	0.0503807	0.2015067	0.3917255	0.4824603	0.4988550
-0.5	0.0000409	0.0020264	0.0312570	0.1666667	0.3726018	0.4792763	0.4986910
-0.7	0.0000011	0.0002886	0.0131770	0.1265930	0.3545218	0.4775386	0.4986513

(b) $x_1=1$

ρ_{12} \ x_2	-3	-2	-1	0	1	2	3
0.7	0.0013498	0.0227465	0.1581433	0.4868229	0.7666683	0.8370300	0.8412928
0.5	0.0013481	0.0226032	0.1548729	0.4687429	0.7452036	0.8318608	0.8410314
0.3	0.0013241	0.0219058	0.1483382	0.4496193	0.7281473	0.8272825	0.8406612
0.1	0.0012270	0.0203167	0.1390450	0.4303256	0.7140097	0.8236409	0.8403322
0.	0.0011357	0.0191407	0.1334838	0.4206724	0.7078610	0.8222040	0.8402090
-0.1	0.0010125	0.0177038	0.1273350	0.4110192	0.7022997	0.8210280	0.8401177
-0.3	0.0006835	0.0140622	0.1131974	0.3917255	0.6930065	0.8194389	0.8400205
-0.5	0.0003133	0.0094839	0.0961411	0.3726018	0.6864718	0.8187415	0.8399965
-0.7	0.0000519	0.0043147	0.0746764	0.3545218	0.6832014	0.8185982	0.8399949

(c) $x_1=-1$

ρ_{12} \ x_2	-3	-2	-1	0	1	2	3
0.7	0.0012979	0.0184353	0.0839788	0.1454782	0.1581433	0.1586517	0.1586552
0.5	0.0010365	0.0132662	0.0625140	0.1273982	0.1548729	0.1585084	0.1586536
0.3	0.0006663	0.0086878	0.0454578	0.1082745	0.1483382	0.1578109	0.1586296
0.1	0.0003373	0.0050462	0.0313202	0.0889808	0.1390450	0.1562218	0.1585324
0.	0.0002141	0.0036094	0.0251714	0.0793276	0.1334838	0.1550458	0.1584411
-0.1	0.0001228	0.0024334	0.0196102	0.0696744	0.1273350	0.1536090	0.1583179
-0.3	0.0000256	0.0008443	0.0103170	0.0503807	0.1131974	0.1499674	0.1579889
-0.5	0.0000016	0.0001468	0.0037823	0.0312570	0.0961411	0.1453890	0.1576187
-0.7	0.0000000	0.0000035	0.0005119	0.0131770	0.0746764	0.1402199	0.1573573

random variables ($\rho_{12}=0$) for any fixed value of x_2, while those in case of negative values ($\rho_{12}<0$) are less than those in case of $\rho_{12}=0$. The numerical values are listed in Table 1, which illustrates the above statement quantitatively.

In order to examine the contribution of the moment terms of various order, partial sums of the series (ΔP_{2N}) are calculated and plotted in Fig. 3 for the cases of $\rho_{12}=0.5$ and $\rho_{12}=0.9$. The contributions of the second order terms are dominant for both cases. As N becomes large, ΔP_{2N} becomes small, and thus its contribution on the value of the *p.d.f.* becomes also small. It should be noted here that the effects of the higher order terms are dependent on the values of the correlation coefficient ρ_{12} as shown in the figure.

The computer processing times are plotted in Fig. 4 against the moment terms retained to calculate the *p.d.f.*. The processing time becomes large as the order of the moment terms retained is increased. The computations are processed by the use of TOSBAC-5600 MODEL-120 computer system at the Computer Center of the University of Osaka Prefecture.

In order to attain computational accuracy, the moment terms should be taken to the highest possible order, which requires a large computer processing time. Hence the value of N must be selected considering a compromise between the accuracy and the

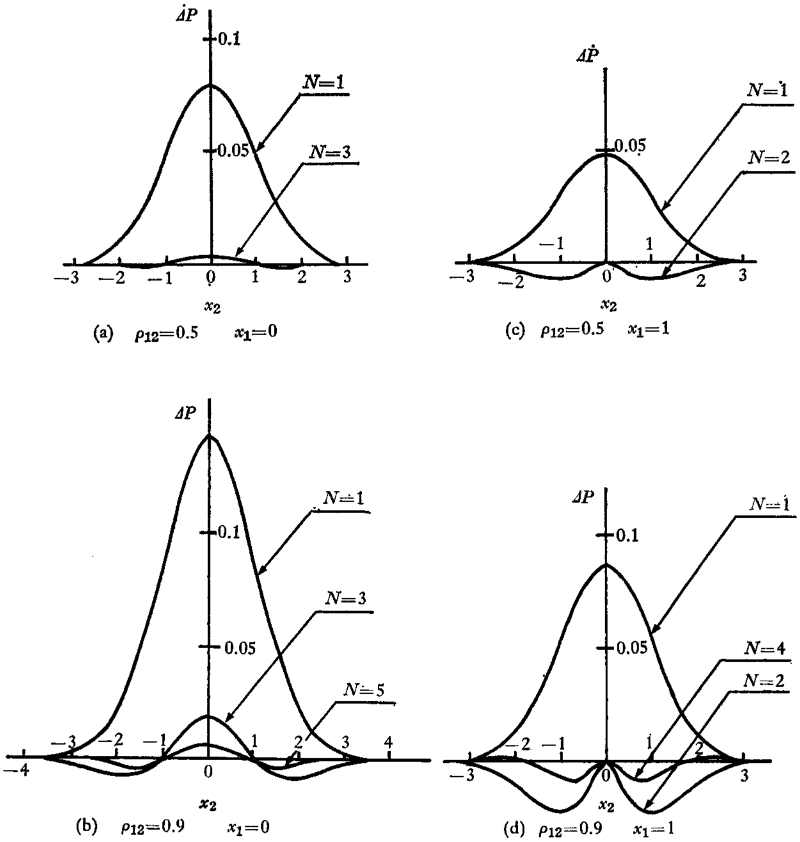

Fig. 3. Contribution of ΔP_{2N}.

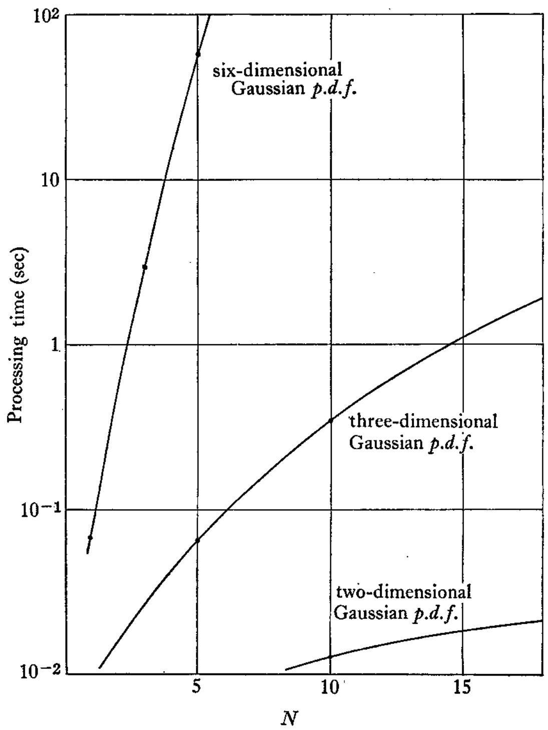

Fig. 4. Computer processing time against moment terms.

Table 2. The numerical values of ΔP_{2N} (1, 1).

N \ ρ_{12}	0.1	0.3	0.5	0.7	0.9
1	0.5854981E-02	0.1756494E-01	0.2927490E-01	0.4098487E-01	0.5269483E-01
2	0.2927491E-03	0.2634741E-02	0.7318726E-02	0.1434470E-01	0.2371267E-01
3	0.	0.	0.	0.	0.
4	0.9758305E-06	0.7904225E-04	0.6098939E-03	0.2342968E-02	0.6402421E-02
5	0.1951661E-07	0.4742535E-05	0.6098939E-04	0.3280155E-03	0.1152436E-02
6	0.2927492E-08	0.2134141E-05	0.4574204E-04	0.3444163E-03	0.1555788E-02
7	0.2973960E-09	0.6504048E-06	0.2323405E-04	0.2449183E-03	0.1422435E-02
8	0.5808517E-11	0.3810966E-07	0.2268951E-05	0.3348492E-04	0.2500374E-03
9	0.2811322E-11	0.5533523E-07	0.5490860E-05	0.1134469E-03	0.1089163E-02
10	0.1264966E-14	0.7469494E-10	0.1235318E-07	0.3573213E-06	0.4410660E-05
11	0.2168889E-13	0.3842119E-08	0.1059027E-05	0.4288598E-04	0.6806197E-03
12	0.1070880E-15	0.5691089E-10	0.2614450E-07	0.1482234E-05	0.3024477E-04
13	0.1455078E-15	0.2319863E-09	0.1776217E-06	0.1409810E-04	0.3698610E-03
14	0.3763462E-17	0.1800051E-10	0.2297033E-07	0.2552464E-05	0.8609584E-04
15	0.8532704E-18	0.1224349E-10	0.2603972E-07	0.4050949E-05	0.1756806E-03
16	0.6167317E-19	0.2654825E-11	0.9410568E-08	0.2049577E-05	0.1142815E-03
17	0.4220690E-20	0.5450601E-12	0.3220127E-08	0.9818600E-06	0.7038911E-04
18	0.7593919E-21	0.2942036E-12	0.2896846E-08	0.1236602E-05	0.1139805E-03
19	0.1578400E-22	0.1834511E-13	0.3010555E-08	0.1799199E-06	0.2132181E-04
20	0.	0.2783640E-13	0.7613563E-09	0.6370134E-06	0.9705947E-04

Table 3. Relative errors $|(P^*-P)/P^*|$ in the calculation of $P(-1, -1)$

N \ ρ_{12}	0.1	0.3	0.5	0.7	0.9
0	0.19631E-00	0.44626E-00	0.59734E-00	0.70026E-00	0.78204E-00
1	0.93782E-02	0.59865E-01	0.12905E-00	0.21223E-00	0.32577E-00
2	0.31289E-04	0.19057E-02	0.11980E-01	0.41418E-01	0.12045E-00
3	0.31289E-04	0.19057E-02	0.11980E-01	0.41418E-01	0.12045E-00
4	0.	0.16674E-03	0.22249E-02	0.13519E-01	0.65017E-00
5	0.63856E-06	0.62475E-04	0.12493E-02	0.96137E-02	0.55039E-01
6	same	0.15618E-04	0.51748E-03	0.55125E-02	0.41567E-01
7		0.13199E-04	0.14588E-03	0.25961E-02	0.29250E-01
8		0.43996E-06	0.10957E-03	0.21974E-02	0.27086E-01
9		0.87993E-06	0.21755E-04	0.84651E-03	0.17655E-01
10		same	0.21595E-04	0.84223E-03	0.17617E-01
11			0.46389E-05	0.33162E-03	0.11723E-01
12			0.41590E-05	0.31388E-03	0.11462E-01
13			0.14396E-05	0.14610E-03	0.82595E-02
14			0 95978E-06	0.11562E-03	0.75140E-02
15			0.63985E-06	0.67397E-04	0.59927E-02
16			same	0.42986E-04	0.50030E-02
17				0.31317E-04	0.43934E-02
18				0.16551E-04	0.34072E-02
19				0.14408E-04	0.32219E-02
20				0.69064E-05	0.23820E-02

P^*; Value given in the statistical tables
P ; Value calculated by the present authors

computer processing time. Table 2 also illustrates the contribution of the higher order terms for various values of the correlation coefficient. From the table, it is seen that the effects of the higher order terms can not be neglected as the value of the correlation coefficient approaches to unity.

Further to evaluate the effect of the moment terms on the resulting probability, statistical tables[5] are referred. The relative errors are tabulated in Table 3 as N is changed for the case of $P(-1, -1)$. In the table, P^* and P correspond to the values given in the statistical tables and those calculated by the present authors, respectively. It is known that the values of the two dimensional Gaussian $p.d.f$ calculated by the proposed method are acceptable when N is taken to be 10 except the cases where $|\rho_{12}| \geq 0.7$.

Next consider the Gaussian $p.d.f.$ whose dimensions are greater than two. Some numerical results are given in Tables 4 and 5 for three- and six-dimensional cases. In Table 4, the moment terms are retained to $N=20$ as the highest order and the resulting probabilities are compared. The values of the $p.d.f.$ are converged to constant values for the cases of $\rho_{ij}=0.1,0.3$ and 0.5, while those for the cases of $\rho_{ij}=0.7$ and 0.9 oscillate. Consequently, the selection of N is an important subject in the future, considering the convergence condition, accumulation of error, etc. for large values of ρ_{ij}. The calculated values of the six-dimensional Gaussian $p.d.f.$ are listed in Table 5. Concerning the values in the above tables, there are no standard references available and thus evaluation

Table 4. Effect of moment terms retained on the values of the three-dimensional Gaussian $p.d.f.$

(a) $P(1, 1, 1)$

N \ ρ_{ij}	0.1	0.3	0.5	0.7	0.9
0	0.5955551	0.5955551	0.5955551	0.5955551	0.5955551
1	0.6103333	0.6398896	0.6694460	0.6990023	0.7285587
2	0.6106472	0.6427146	0.6772932	0.7143829	0.7539836
3	0.6106330	0.6423321	0.6755223	0.7095235	0.7436556
4	0.6106376	0.6427037	0.6783898	0.7205396	0.7737584
5	0.6106372	0.6426124	0.6772156	0.7142242	0.7515701
6	0.6106373	0.6426447	0.6779066	0.7194271	0.7750727
7	0.6106373	0.6426404	0.6777549	0.7178285	0.7657882
8	0.6106373	0.6426394	0.6776951	0.7169446	0.7591881
9	0.6106373	0.6426413	0.6778844	0.7208572	0.7967513
10	0.6106373	0.6426400	0.6776692	0.7146316	0.7199052
11	0.6106873	0.6426408	0.6778719	0.7228404	0.8501829
12	0.6106373	0.6426404	0.6777091	0.7136117	0.6618716
13	0.6106373	0.6426406	0.6778252	0.7228228	0.9035239
14	0.6106373	0.6426405	0.6777559	0.7151216	0.6437564
15	0.6106373	0.6426405	0.6777846	0.7195847	0.8373125
16	0.6106373	0.6426405	0.6777885	0.7204345	0.8846946
17	0.6106373	0.6426405	0.6777608	0.7120072	0.2805450
18	0.6106373	0.6426405	0.6778039	0.7303910	0.1975023
19	0.6106373	0.6426405	0.6777527	0.6998016	0.1650033
20	0.6106373	0.6426405	0.6778060	0.7444260	0.5149225

Table 4.
(b) $P(-1, -1, -1)$

N \ ρ_{ij}	0.1	0.3	0.5	0.7	0.9
0	0.3993589E-02	0.3993589E-02	0.3993589E-02	0.3993589E-02	0.3993589E-02
1	0.6780360E-02	0.1235390E-01	0.1792744E-01	0.2350098E-01	0.2907452E-01
2	0.7344718E-02	0.1743313E-01	0.3203641E-01	0.5115456E-01	0.7478757E-01
3	0.7358886E-02	0.1781565E-01	0.3380732E-01	0.5601395E-01	0.8511556E-01
4	0.7357225E-02	0.1768113E-01	0.3276942E-01	0.5202676E-01	0.7422011E-01
5	0.7357659E-02	0.1778667E-01	0.3412664E-01	0.5932620E-01	0.9986564E-01
6	0.7357624E-02	0.1776083E-01	0.3357286E-01	0.5515652E-01	0.8103047E-01
7	0.7357627E-02	0.1776703E-01	0.3379422E-01	0.5748990E-01	0.9458227E-01
8	0.7357627E-02	0.1776815E-01	0.3386092E-01	0.5847423E-01	0.1019324E-01
9	0.7357627E-02	0.1776641E-01	0.3368802E-01	0.5490200E-01	0.6763671E-01
10	0.7357627E-02	0.1776771E-01	0.3390328E-01	0.6112862E-01	0.1444961E-01
11	0.7357627E-02	0.1776699E-01	0.3370375E-01	0.5304845E-01	0.1626027E-01
12	0.7357627E-02	0.1776734E-01	0.3386661E-01	0.6228164E-01	0.2046623E-00
13	0.7357627E-02	0.1776719E-01	0.3375110E-01	0.5311281E-01	−0.3588046E-01
14	0.7357627E-02	0.1776724E-01	0.3382047E-01	0.6082173E-01	0.2241453E 00
15	0.7357627E-02	0.1776723E-01	0.3379186E-01	0.5637075E-01	0.3111632E 01
16	0.7357627E-02	0.1776723E-01	0.3378799E-01	0.5552713E-01	−0.1592300E-01
17	0.7357626E-02	0.1776723E-01	0.3381563E-01	0.6395738E-01	0.5884379E 00
18	0.7357627E-02	0.1776723E-01	0.3377258E-01	0.4557728E-01	−0.1105698E 01
19	0.7357627E-02	0.1776723E-01	0.3382376E-01	0.7616715E-01	0.2519422E 01
20	0.7357627E-02	0.1776723E-01	0.3377043E-01	0.3154468E-01	−0.4279545E 01

Table 5. Effect of moment terms retained on the values of the six-dimensional Gaussian *p.d.f.*
(a) $P(1, 1, 1, 1, 1, 1)$

N \ ρ_{ij}	0.05	0.1	0.3	0.5
0	0.3546859	0.3546859	0.3546859	0.3546859
1	0.3766889	0.3986920	0.4867041	0.5747163
2	0.3762464	0.3969218	0.4707725	0.5304617
3	0.3762782	0.3971764	0.4776474	0.5622902
4	0.3762768	0.3971535	0.4757912	0.5479672
5	0.3762768	0.3971539	0.4758808	0.5491194

(b) $P(-1, -1, -1, -1, -1, -1)$

N \ ρ_{ij}	0.05	0.1	0.3	0.5
0	0.1594875E-04	0.1594875E-04	0.1594875E-04	0.1594875E-04
1	0.4377180E-04	0.4377180E-04	0.1828870E-03	0.2941792E-03
2	0.6266176E-04	0.6266176E-04	0.8629257E-03	0.2183176E-02
3	0.6856753E-04	0.6856753E-04	0.2138572E-02	0.8088947E-02
4	0.6922783E-04	0.6922783E-04	0.2994316E-02	0.1469190E-01
5	0.6918482E-04	0.6918482E-04	0.2659862E-02	0.1039080E-01

Table 6. Effect of dimension on computer processing time

N \ dimension	2	3	6
1	0.00484 sec	0.00938 sec	0.06795 sec
3	0.00530	0.02700	2.92925
5	0.00791	0.06623	55.85778
10	0.01336	0.35189	....
20	0.02580	2.65263	....

of the accuracy has not been made in this paper.

Finally the effect of dimensions on the computer processing time are demonstrated in Table 6, which shows that the processing time swells abruptly as the dimensions become large with the moment terms retained to higher order.

5. Conclusion

Applying the Hermite polynomial expansion method, an algorithmic procedure has been developed for calculating multi-dimensional Gaussian probability distribution functions of arbitrary dimensions taking account of the moment terms up to an arbitrary order. Numerical examples are presented to demonstrate the applicability of the proposed method, and are discussed the effects of correlation coefficients, the order of moment terms and the dimensions of the *p.d.f.* on the resulting probability and the computer processing time. Comparison of the proposed method with others[6),7)] is being performed and will be reported in the near future.

Acknowledgements

The authors wish to express their sincere appreciation to Professor Emeritus G. OKUNO and Professor K. MATSUOKA for their encouragement, to Professor K. TAGUCHI for his stimulating remarks and to Professor T. TSUMURA for his valuable advice. This work was in part supported by a science research fund of the Ministry of Education of Japan, No. C-055256.

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