Number of Hamiltonian Circuits in Basic Series of Incomplete Graphs

Number of Hamiltonian Circuits in Basic Series of Incomplete Graphs

Tamotsu KASAI*, Hiroshi SHINTANI** and Mitsuru IMAI***

(Received November 15, 1971)

The problem of counting Hamiltonian circuits in incomplete graphs made by removing some branches from a complete graph is dealt with and a method for finding the number of Hamiltonian circuits is described. The concept which was named as the "*inversion of branches*" by the authors is introduced and then the formulas for the number of Hamiltonian circuits in six basic series of incomplete graphs are derived. Some applications of the method in this paper to the incomplete graphs, which do not belong to the basic series, are also shown as the examples.

1. Introduction

A sequence of branches in a given graph is called a *path* when no branch appears more than once in it. If the path forms a loop, this cyclic path is called a *circuit*. *Hamilton circuit* is defined by the property of being cyclic path with respect to the nodes and if a circuit passes through every node in the finite connected graph, this circuit is called *Hamiltonian*. On the other hand, there is a sequence of branches which is called *Euler path*. This path is characterized by the property of being cyclic path with respect to the branches and the every branch in the graph appears just once in it.

In spite of the similarity of the definitions of Euler paths and Hamiltonian circuits, the theories for the two concepts have little in common. For Euler paths, it is easy to establish a criterion for their existence, that is, if and only if a given finite graph is connected and the *local degree*, which is defined as the number of branches at every node of the graph, is even number at all the nodes, that graph has some Euler paths.[1)] However, for Hamiltonian circuits no such general rule is known and even in a specific graph it may be difficult to decide whether such a circuit can be found.

Since Hamiltonian circuits contain all the nodes in a given graph and pass through every node just once, by eliminating the branches which return to the nodes already passed in the process of tracing the branches in order, or by making the system of operation[2)] embracing such a process, we can find Hamiltonian circuits themselves in that graph at the end of the operation. However, when a given graph has a complex form, these works will take plenty of time.

The purpose of this paper is to give a general method for finding the number of Hamiltonian circuits in incomplete graphs. Computing the number of Hamiltonian circuits in a given graph in advance will give an important aim in connection with the problem of the existence of their circuits in that graph.

* Department of Electrical Engineering, College of Engineering.
** Graduate Student, Department of Electrical Engineering, College of Engineering.
*** Undergraduate, Department of Electrical Engineering, College of Engineering.

The fundamental concept, which is used throghout the paper, is an idea named "*inversion of branches*". By using this concept, the formulas for the number of Hamiltonian circuits in six basic series of incomplete graphs, which are called *r*-, *p*-, *c*-, *m*-, *s*- and *h*-*Series*, are derived. Furthermore, in order to show that the method in this paper can also be applied to the incomplete graphs, which do not belong to the basic series just mentioned above, some examples are given.

2. Number of Hamiltonian circuits in a complete graph

The complete graphs having nodes over three in number are sure to have Hamiltonian circuits. We now consider a complete graph of N nodes. Since the local degree at every node of this graph is $N-1$, by tracing any one of the $N-1$ branches which are incidental to the first node, that is, the starting point, we can arrive at the second node. The second node has $N-2$ branches which are connected to the other nodes except the first node, and in general, for i-th node, the number of such the branches becomes $N-i$, where $i=1, 2, \cdots, N-1$. Consequently, by tracing any one of the $N-i$ branches, we can arrive at the $i+1$-th node from i-th node. Thus, the running number of the circuits passing through every node and returning to the first node is given by $(N-1)!$. Hereupon, we must pay attention to the fact that the same Hamiltonian circuit is computed twice into $(N-1)!$. In consequence of this fact, the number of Hamiltonian circuits in a complete graph of N nodes is given by

$$(N-1)!/2. \tag{1}$$

3. Inversion of branches[3)]

In this chapter, we consider Hamiltonian circuits containing some specific branches in a complete graph. Fig. 1 shows two types of Hamiltonian circuits containing α specific branches connected in series. These two types are sure to exist as a pair in the complete graph of N nodes.

Comparing graphically the two Hamiltonian circuits shown in Fig. 1, we can observe that one of them is obtained by connecting inversely right and left of the α branches after taking off two branches which are connectected to the nodes 1 and $\alpha+1$ in the other. Such an inversion does not carry out in practice, but it is used as an important concept in

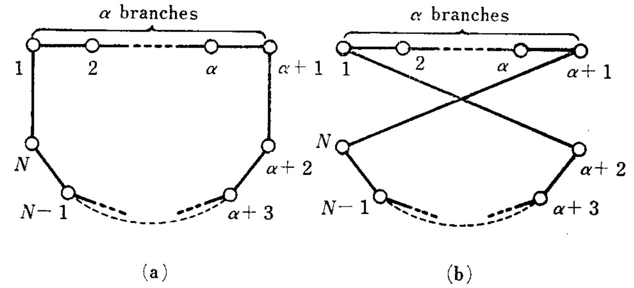


Fig. 1 Two types of Hamiltonian circuits containing the specific branches.

this paper and we will give it the name of "*inversion of branches*".

From the above consideration, the number of Hamiltonian circuits containing α specific branches connected in series in a complete graph of N nodes can be obtained by the following equation:

$$2\{(N-1-\alpha)!/2\}=(N-1-\alpha)!. \tag{2}$$

In the left side of Eq. (2), the multiplier 2 is based on the concept of "inversion of branches" and $(N-1-\alpha)!/2$, as we can understand from Formula (1), gives the number of Hamiltonian circuits in a complete graph of $N-\alpha$ nodes. Consequently, Eq. (2) shows that the number of Hamiltonian circuits containing α specific branches in a complete graph is obtained by counting that in the complete graph of $N-\alpha$ nodes made by putting together $\alpha+1$ nodes incident to the α branches. For example, if we consider the Hamiltonian circuits containing two specific branches $(\alpha=2)$ connected in series in the complete graph of six nodes, the procedure in accordance with Eq. (2) becomes as shown in Fig. 2.

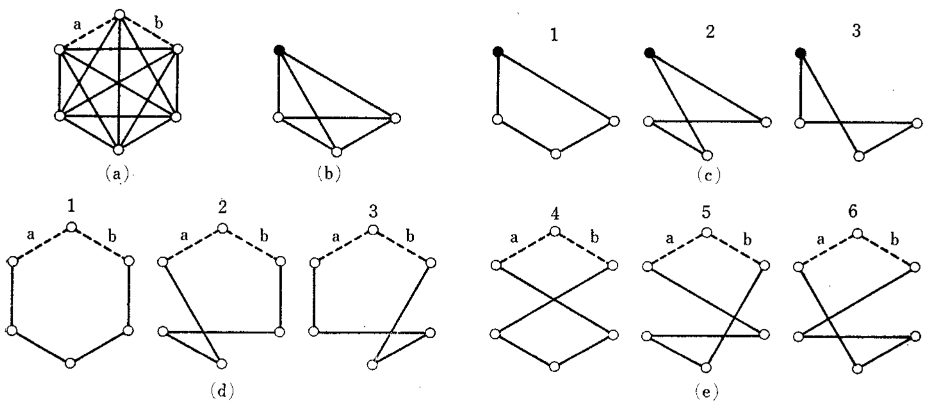


Fig. 2 Procedure for finding the Hamiltonian circuits containing two specific branches in the complete graph of six nodes.

Fig. 2(a) is the complete graph of six nodes, with the two specific branches shown in dotted lines, Fig. 2(b) is the complete graph having the same number of nodes as the value of $N-\alpha$ and Fig. 2(c) shows Hamiltonian circuits in the graph of Fig. 2(b). The number of these circuits equals to $(N-1-\alpha)!/2$. In each of the circuits shown in Fig. 2 (c), by separating a marked node into two nodes with one branch, respectively, and by inserting the specific branches stated above between the two nodes, three Hamiltonian circuits as shown in Fig. 2(d) are obtained and the number equals to that of the circuits of Fig. 2 (c). Furthermore, by using the concept of "inversion of branches", the circuits of Fig. 2(e) are obtained from those of Fig. 2(d). The graphs of Fig. 2(d) and (e) are Hamiltonian circuits containing both branches a and b in the complete graph of Fig. 2(a), and the number of these Hamiltonian circuits agrees with the value obtained from Eq. (2).

By expanding the above consideration, we can find the number of Hamiltonian circuits containing β sets of branches as shown in Fig. 3 in a complete graph. In this case, it is evident that the number of combinations causing by "inversion of branches" is given by 2^{β}.

Consequently, in much the same way as the preceding consideration, the number of Hamiltonian circuits containing the sets of branches, shown in Fig. 3, is given by

$$2^{\beta}[\{N-1-(\alpha_1+\alpha_2+\cdots+\alpha_{\beta})\}!/2]=2^{\beta-1}\cdot(N-1-\sum_{i=1}^{\beta}\alpha_i)!. \tag{3}$$

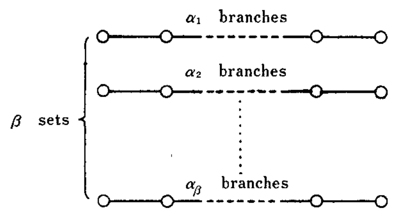


Fig. 3 Sets of branches connected in series.

4. Patterns of the removed branches belonging to Hamiltonian circuits in a complete graph

Since any incomplete graph is derived by removing some branches from a complete graph, the number of Hamiltonian circuits in a given incomplete graph can be found theoretically by obtaining the difference between the following values:

(1) Number of Hamiltonian circuits in a complete graph having the same number of ndoes as the incomplete graph.

(2) Number of Hamiltonian circuits containing the branches to be removed from the complete graph to make the incomplete graph.

The value of the former is obtained from Formula (1). In order to obtain the value of the later, we now consider the forms of branches belonging to the *complementary graph** of the given incomplete graph and belonging to Hamiltonian circuits in the original complete graph. Let k be the number of such the branches. The forms of the removed branches belonging to Hamiltonian circuits in the origial complete graph are determined by the values of k, and the patterns to simple values of k become as shown in Fig. 4. As is evident from Fig. 4, β has the values from 1 to k.

Each of the patterns, shown in Fig. 4, is made by combining some of the branches to be removed from a complete graph. Consequently, if the number of such the combinations is calculated, the number of Hamiltonian circuits in a given incomplete graph can be obtained from Formula (1) and Eq. (3).

5. Formulas for the number of Hamiltonian circuits in basic series of incomplete graphs

The six basic series of incomplete graphs dealing with in this chapter are called r-, p-, c-, m-, s- and *h-Series*,[4)~6)] respectively. The incomplete graphs belonging to these series are made by removing the graphs as shown in Fig. 5 from a complete graph. The

* The graph to be removed from a complete graph to make a given incomplete graph is called complementary graph.

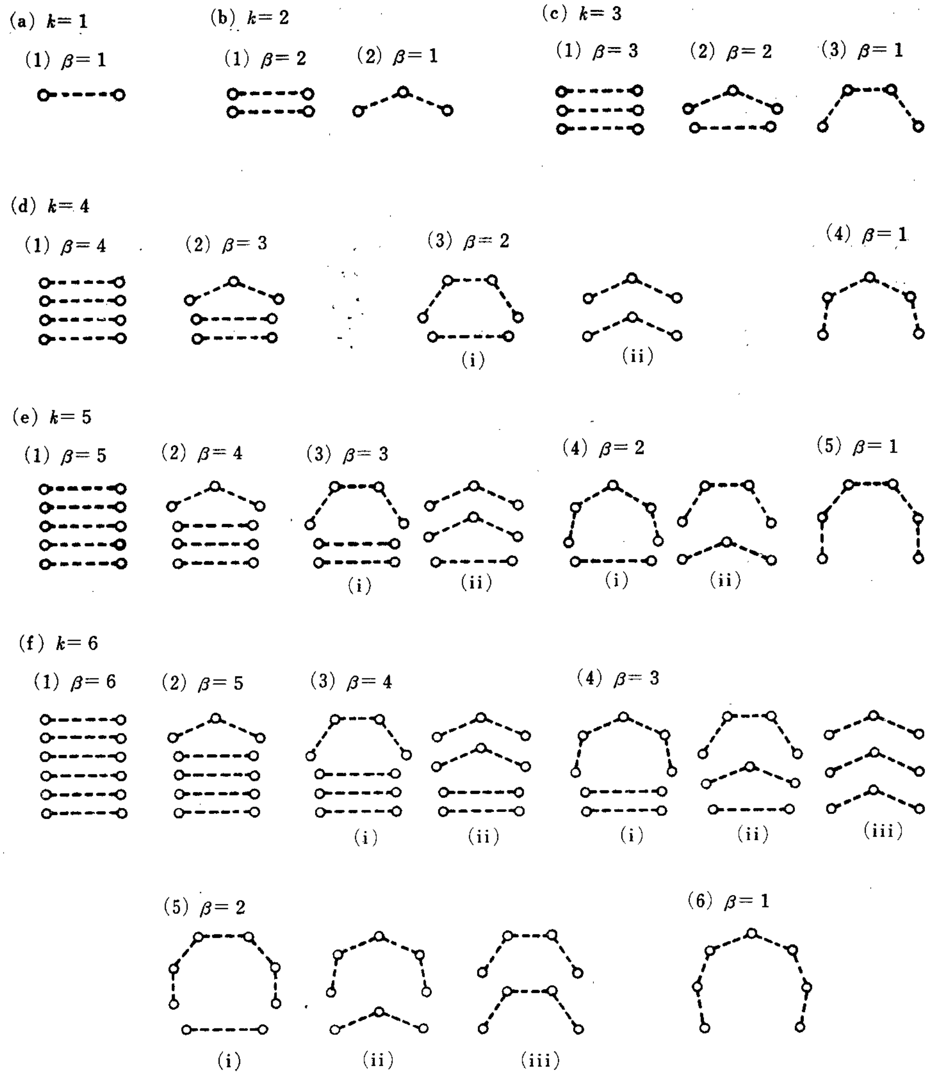


Fig. 4 Patterns of the removed branches belonging to Hamiltonian circuits in a complete graph.

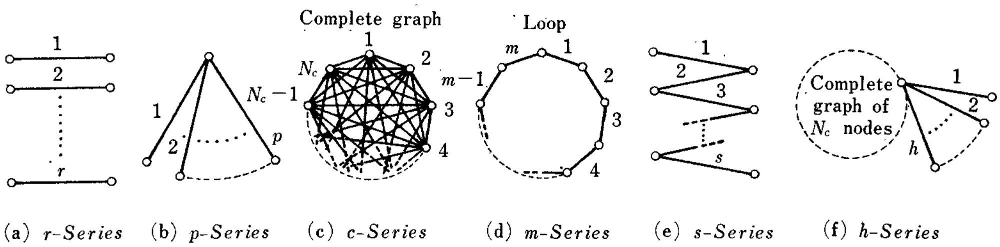


Fig. 5 Forms of branches to be removed from a complete graph in six basic series of incomplete graphs.

formulas for the number of Hamiltonian circuits in these six series of incomplete graphs are derived by obtaining the number of selecting k branches, which form the patterns in Fig. 4, from the branches belonging to each of the graphs shown in Fig. 5.

5.1 Formula for the number of Hamiltonian circuits in *r-Series*

The r branches, shown in Fig. 5(a), to be removed from a complete graph make the r patterns of $k=1,2,\cdots,r$ and $\beta=k$, shown in Fig. 4, as the forms which are contained within Hamiltonian circuits in the original complete graph. For convenience of the consideration, we now consider an incomplete graph of $N=5$ and $r=2$. Fig. 6 shows Hamiltonian circuits ccntaining either or both of the two dotted-line branches which were removed from the complete graph.

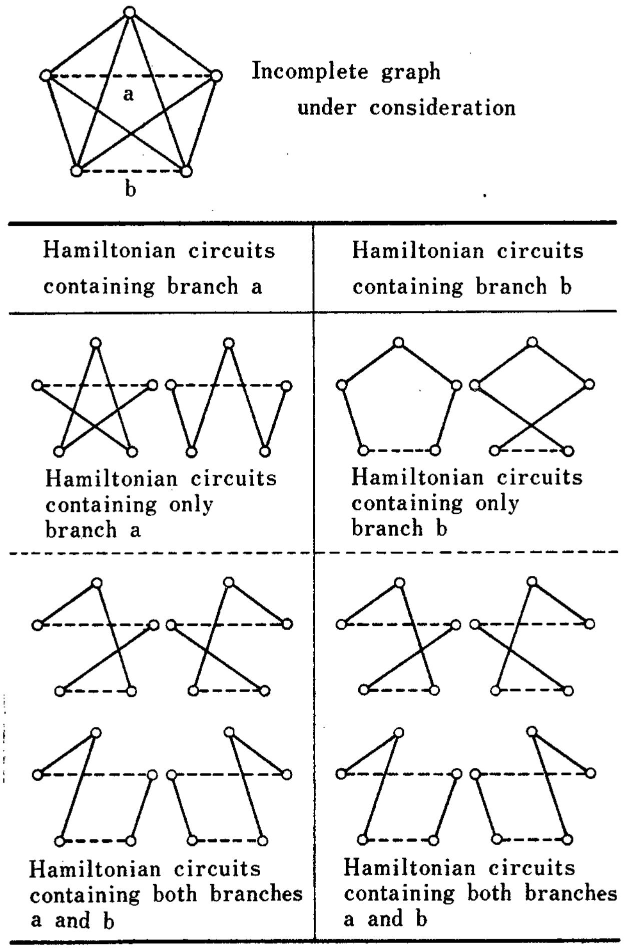


Fig. 6 Hamiltonian circuits containing either or both of the two removed branches.

First the following values are obtained from Eq. (3):

(1) The number of Hamiltonian circuits containing branch a or containing brahch b is

$$(N-2)!=3!. \tag{4}$$

(2) The number of Hamiltonian circuits containing both branches a and b is

$$2\cdot(N-3)!=2\cdot2!. \tag{5}$$

(3) The number of Hamiltonian circuits containing only branch a or containing only branch b is

$$(N-2)!-2\cdot(N-3)!=3!-2\cdot 2!. \tag{6}$$

Consequently, from Formula (1), Eqs. (4), (5) and (6), the number of Hamiltonian circuits in incomplete graph under consideration is given by

$$\begin{aligned}(N-1)!/2-2\{(N-2)!-2\cdot(N-3)!\}-2\cdot(N-3)!\\ =(N-1)!/2-2\cdot(N-2)!+2\cdot(N-3)!\\ =4!/2-2\cdot 3!+2\cdot 2!,\end{aligned} \tag{7}$$

and becomes 4.

Eq. (7) shows that there is no necessity for calculating expressly the value of Eq. (6). In the last equation of Eq. (7), the first term expresses the number of Hamiltonian circuits in the complete graph of $N=5$ and the second term gives the number of all the circuits, shown in Fig. 6. The second term, however, has doubly the Hamiltonian circuits containing both branches a and b, as shown in Fig. 6. Consequently, it is necessary to add the third term representing the number of such the Hamiltonian circuits. With similar consideration, we can find generally the formula for the number of Hamiltonian circuits in *r-Series* of incomplete graphs.

Since the number of selecting k branches out of r branches to be removed from a complete graph is given by ${}_rC_k$, where $k=1, 2, \cdots, r$, if the expression for the number of Hamiltonian circuits, H_r, is written in the same form as Eq. (7), it becomes as follows:

$$\begin{aligned}H_r=(N-1)!/2-{}_rC_1\cdot(N-2)!+{}_rC_2\cdot 2\cdot(N-3)!-{}_rC_3\cdot 2^2\cdot(N-4)!+\cdots\\ \cdots+(-1)^r\cdot{}_rC_r\cdot 2^{r-1}\cdot(N-1-r)!.\end{aligned} \tag{8}$$

Each term in Eq. (8) has the similar meaning as that in Eq. (7). Arranging Eq. (8), we obtain the following formula:

$$H_r=\sum_{k=0}^{r}\{(-1)^k\cdot 2^{k-1}\cdot{}_rC_k\cdot(N-1-k)!\}, \qquad 0<2r\leqq N. \tag{9}$$

5.2 Formula for the number of Hamiltonian circuits in *p-Series*

The p branches, shown in Fig. 5(b), to be removed from a complete graph make the two patterns of $k=1, 2$ and $\beta=1$, shown in Fig. 4, as the forms which are contained within Hamiltonian circuits in the original complete graph. Fig. 7(a) shows p branches to be removed from a single node A of a complete graph and $N-1-p$ remaining branches, and Fig. 7(b) shows two kinds of Hamiltonian circuits containing the patterns just mentioned above in the original complete graph.

The numbers of selecting one and two branches out of p branches are given by p and ${}_pC_2=p(p-1)/2$, respectively. Consequently, by using these values and the concept of "inversion of branches", we can find the number of Hamiltonian circuits in *p-Series* of incomplete graphs, which becomes as follows:

$$\begin{aligned}H_p=(N-1)!/2-p\cdot(N-2)!+p(p-1)\cdot(N-3)!/2\\ =(N-1-p)(N-2-p)\cdot(N-3)!/2.\end{aligned} \tag{10}$$

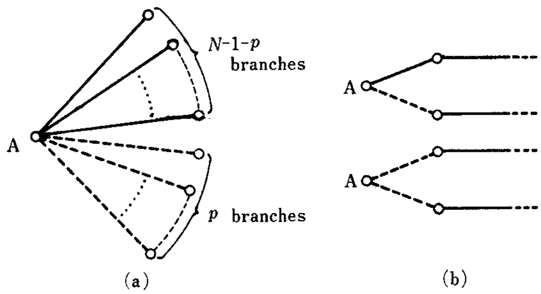


Fig. 7 State of branches at single node A and two kinds of Hamiltonian circuits containing the branches to be removed.

From Formula (10), it is evident that the Hamiltonian circuits exist when $N \geqq 3$ and $N-3 \geqq p$.

5.3 Formula for the number of Hamiltonian circuits in *c-Series*

The incomplete graphs belonging to this series are derived by removing another complete graph of $N_c(<N)$ nodes, shown in Fig. 5(c), from a complete graph of N nodes. Each of patterns, shown in Fig. 4, is made by combining some of the branches belonging to the removed complete graph and the number of their combinations can be found as follows:

$$k! \cdot {}_{N_c}C_{2k+1-\gamma} \cdot {}_{2k+1-\gamma}C_k \cdot {}_{k-1}C_{\gamma-1}/2^{k+1-\gamma}, \qquad 1 \leqq \gamma \leqq N_c-1. \tag{11}$$

In Eq. (11), the value for $\gamma=1$ expresses the number of combinations for making the patterns (1) in Fig. 4, the value for $\gamma=2$ corresponds to the patterns (2) and so on.

Thus, the number of Hamiltonian circuits in *c-Series* of incomplete graphs is arranged as the following equation:

$$H_c = \frac{1}{2} \sum_{\gamma=0}^{N_c-1} \left[\sum_{k=\gamma}^{l_c} \{(-1)^k \cdot F_c(\gamma, k) \cdot (N-1-k)!\} \right], \qquad N_c \geqq 2, \tag{12}$$

where

$$F_c(\gamma, k) = \begin{cases} 1 & \text{for } \gamma=0, \\ k! \cdot {}_{N_c}C_{2k+1-\gamma} \cdot {}_{2k+1-\gamma}C_k \cdot {}_{k-1}C_{\gamma-1} & \text{for } \gamma \geqq 1, \end{cases} \tag{13}$$

$$l_c = \begin{cases} 0 & \text{for } \gamma=0, \\ (N_c-1+\gamma)/2 & \text{for } \gamma \geqq 1 \text{ if one of } N_c \text{ and } \gamma \text{ is odd and the other is even,} \\ (N_c-2+\gamma)/2 & \text{for } \gamma \geqq 1 \text{ if both } N_c \text{ and } \gamma \text{ are odd or even,} \end{cases} \tag{14}$$

and the value of Formula (12) for $\gamma=0$ expresses the number of Hamiltonian circuits in the complete graph of N nodes. This matter is also the same in the formulas to be given in the succeeding sections.

The calculated results of Formula (12) to some simple values of N_c are shown in Table 1. From this table, we observe that Formula (12) can be rewritten in the simple form as follows:

$$H_c = \frac{1}{2} \cdot (N-N_c)! \cdot \prod_{\lambda=1}^{N_c-1} (N-N_c-\lambda), \qquad N_c \geqq 2. \tag{15}$$

Table 1 Expressions for H_G

N_G	H_G
2	$(N-2)!\cdot(N-3)/2$
3	$(N-3)!\cdot(N-4)(N-5)/2$
4	$(N-4)!\cdot(N-5)(N-6)(N-7)/2$
5	$(N-5)!\cdot(N-6)(N-7)(N-8)(N-9)/2$
6	$(N-6)!\cdot(N-7)(N-8)(N-9)(N-10)(N-11)/2$
7	$(N-7)!\cdot(N-8)(N-9)(N-10)(N-11)(N-12)(N-13)/2$
⋮	⋮

5.4 Formula for the number of Hamiltonian circuits in *m-Series*

The branches to be removed from a complete graph to make the incomplete graphs belonging to *r*- and *p-Series* do not form the circuits by themselves. In the incomplete graphs belonging to *c-Series*, such the branches form some circuits by themselves, but there are no circuits which become Hamiltonian in the original complete graph. On the other hand, the set of branches, shown in Fig. 5(d), to be removed from a complete graph in *m-Series* is clearly a circuit and when $N=m$, it becomes Hamiltonian circuit in the original complete graph. Furthermore, when $N=m\geqq 5$, the incomplete graphs made by removing the m branches from a complete graph have some Hamiltonian circuits. Consequently, in the case of calculating the number of Hamiltonian circuits for $N=m\geqq 5$ in *m-Series*, it is necessary to add one circuit formed by m branches to the result obtained by the same consideration as in *r*-, *p*- and *c-Series*. Since the circuit formed by m branches becomes Hamiltonian in the original complete graph when $N=m$ and does not become Hamiltonian in the original complete graph when $N\neq m$, its number can be expressed by the following notation:

$$\delta_{mN}=\left\{\begin{array}{ll} 0 & \text{for } m\neq N, \\ 1 & \text{for } m=N. \end{array}\right\} \tag{16}$$

The number of combinations for making each of the patterns, shown in Fig. 4, by some of the m branches to be removed from a complete graph is given by

$$m\cdot{}_{m-1-k}C_{k-\gamma}\cdot{}_kC_{\gamma-1}/k, \qquad 1\leqq\gamma\leqq m-1. \tag{17}$$

From Eqs. (16) and (17), the formula for the number of Hamiltonian circuits in *m-Series* of incomplete graphs becomes as follows:

$$H_m=\sum_{\gamma=0}^{m-1}[\sum_{k=\gamma}^{l_m}\{(-1)^k\cdot 2^{k-\gamma}\cdot F_m(\gamma,k)\cdot(N-1-k)!\}]+(-1)^m\cdot\delta_{mN}, \quad 3\leqq m\leqq N, \tag{18}$$

where

$$F_m(\gamma,k)=\left\{\begin{array}{ll} 1/2 & \text{for } \gamma=0, \\ m\cdot{}_{m-1-k}C_{k-\gamma}\cdot{}_kC_{\gamma-1}/k & \text{for } \gamma\geqq 1, \end{array}\right\} \tag{19}$$

$$l_m=\left\{\begin{array}{ll} 0 & \text{for } \gamma=0, \\ (m-1+\gamma)/2 & \text{for } \gamma\geqq 1 \text{ if one of } m \text{ and } \gamma \text{ is odd and the other is even,} \\ (m-2+\gamma)/2 & \text{for } \gamma\geqq 1 \text{ if both } m \text{ and } \gamma \text{ are odd or even,} \end{array}\right\} \tag{20}$$

and the last term in Formula (18) is added when $N=m\geqq 5$.

5.5 Formula for the number of Hamiltonian circuits in *s-Series*

The number of combinations for making each of the patterns, shown in Fig. 4, by some of the s branches, shown in Fig. 5(e), is given by

$$ {}_{s+1-k}C_{k+1-\gamma}\cdot {}_{k-1}C_{\gamma-1}\,, \qquad 1\leqq\gamma\leqq s. \tag{21}$$

From Eq. (21), we obtain the formula for the number of Hamiltonian circuits in *s-Series* of incomplete graphs as

$$H_s=\sum_{\gamma=0}^{s}\Big[\sum_{k=\gamma}^{l_s}\{(-1)^k\cdot 2^{k-\gamma}\cdot F_s(\gamma,k)\cdot(N-1-k)!\}\Big],\qquad 0\leqq s\leqq N-1, \tag{22}$$

where

$$F_s(\gamma,k)=\begin{cases}1/2 & \text{for } \gamma=0,\\ {}_{s+1-k}C_{k+1-\gamma}\cdot{}_{k-1}C_{\gamma-1} & \text{for } \gamma\geqq 1,\end{cases} \tag{23}$$

$$l_s=\begin{cases}0 & \text{for } \gamma=0,\\ (s-1+\gamma)/2 & \text{for } \gamma\geqq1 \text{ if one of } s \text{ and } \gamma \text{ is odd and the other is even,}\\ (s+\gamma)/2 & \text{for } \gamma\geqq 1 \text{ if both } s \text{ and } \gamma \text{ are odd or even.}\end{cases} \tag{24}$$

5.6 Formula for the number of Hamiltonian circuits in *h-Series*

The incomplete graphs belonging to this series are derived by removing another complete graph of N_c nodes having h hinged branches at its one node, shown in Fig. 5(f), from a complete graph. The number of combinations for making each of the patterns, shown in Fig. 4, by some of the branches to be removed from a complete graph is given by

$$\begin{aligned}&\{k!\cdot{}_{N_c}C_{2k+1-\gamma}\cdot{}_{2k+1-\gamma}C_k+2h\cdot(k-1)!\cdot{}_{N_c-1}C_{2k-1-\gamma}\cdot{}_{2k-1-\gamma}C_{k-1}\\&\quad+h(h-1)(\gamma-1)\cdot(k-3)!\cdot{}_{N_c-1}C_{2k-2-\gamma}\cdot{}_{2k-2-\gamma}C_{k-1}\}\cdot{}_{k-1}C_{\gamma-1}/2^{k+1-\gamma},\end{aligned} \tag{25}$$

where the third term in the equation expressing within the brackets is defined for $k\geqq3$. From Eq. (25), we obtain the number of Hamiltonian circuits in *h-Series* of incomplete graphs as

$$H_h=\frac{1}{2}\sum_{\gamma=0}^{\mu}\Big[\sum_{k=\gamma}^{l_h}\{(-1)^k\cdot F_h(\gamma,k)\cdot(N-1-k)!\}\Big],\qquad N_c\geqq2,\ h\geqq0, \tag{26}$$

where

$$\mu=\begin{cases}N_c-1 & \text{for } h=0,\\ N_c & \text{for } h\geqq1,\end{cases} \tag{27}$$

$$F_h(\gamma,k)=\begin{cases}1 & \text{for } \gamma=0,\\ \{k!\cdot{}_{N_c}C_{2k+1-\gamma}\cdot{}_{2k+1-\gamma}C_k+2h\cdot(k-1)!\cdot{}_{N_c-1}C_{2k-1-\gamma}\cdot{}_{2k-1-\gamma}C_{k-1}\\ \quad+h(h-1)(\gamma-1)\cdot(k-3)!\cdot{}_{N_c-1}C_{2k-2-\gamma}\cdot{}_{2k-2-\gamma}C_{k-1}\}\cdot{}_{k-1}C_{\gamma-1} & \text{for } \gamma\geqq1,\end{cases} \tag{28}$$

and l_h depends on N_c and γ, and becomes as follows:

(1) For $h=0$,

$$l_h=\begin{cases}0 & \text{for } \gamma=0,\\ (N_c-1+\gamma)/2 & \text{for } \gamma\geqq1 \text{ if one of } N_c \text{ and } \gamma \text{ is odd and the other is even,}\\ (N_c-2+\gamma)/2 & \text{for } \gamma\geqq1 \text{ if both } N_c \text{ and } \gamma \text{ are odd or even.}\end{cases} \tag{29}$$

(2) For $h=1$,

$$l_h=\begin{cases} 0 & \text{for } \gamma=0, \\ (N_e-1+\gamma)/2 & \text{for } \gamma\geqq 1 \text{ if one of } N_e \text{ and } \gamma \text{ is odd and the other is even,} \\ (N_e+\gamma)/2 & \text{for } \gamma\geqq 1 \text{ if both } N_e \text{ and } \gamma \text{ are odd or even.} \end{cases} \qquad (30)$$

(3) For $h\geqq 2$,

$$l_h=\begin{cases} 0 & \text{for } \gamma=0, \\ N_e/2 & \text{for } \gamma=1 \text{ if } N_e \text{ is even,} \\ (N_e+\gamma)/2 & \text{for } \gamma\geqq 1 \text{ if both } N_e \text{ and } \gamma \text{ are odd or even,} \\ (N_e+1+\gamma)/2 & \text{for } \gamma\geqq 2 \text{ if one of } N_e \text{ and } \gamma \text{ is odd and the other is even.} \end{cases} \qquad (31)$$

When $h=0$, this series reverts to the *c-Series* and Formula (26) agrees with Formula (12). Furthermore, when $N=2$ this series reverts to the *p-Series* and by rewriting $h+1$ to p, Formula (26) agrees with Formula (10) for $p\geqq 1$.

In the above, the formulas for the number of Hamiltonian circuits in six basic series of incomplete graphs have been given. As is evident from these formulas, the number of Hamiltonian circuits in incomplete graphs, H, can also be expressed by the following form:

$$H=(N-1)!/2+\sum_{i=1}^{M}\{(-1)^i\cdot A_i\cdot(N-1-i)!\}, \qquad (32)$$

where the values of M and A_i are determined by the forms of the branches to be removed from a complete graph to derive the incomplete graphs. Hereupon, we must pay attention to the fact that there are some cases where Formula (32) have an additional term as δ_{mN} in *m-Series*. Such an example is also shown in the following chapter.

Tables 2, 3 and 4 show the values of coefficient A_i in Formula (32) representing the number of Hamiltonian circuits in *r-*, *m-* and *s-Series* of incomplete graphs, respectively.

Table 2 Values of A_i in *r-Series*

r	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
1	1									
2	2	2								
3	3	6	4							
4	4	12	16	8						
5	5	20	40	40	16					
6	6	30	80	120	96	32				
7	7	42	140	280	336	224	64			
8	8	56	224	560	896	896	512	128		
9	9	72	336	1008	2016	2688	2304	1152	256	
10	10	90	480	1680	4032	6720	7680	5760	2560	512

Table 3 Values of A_i in *m-Series*

m	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
3	3	3							
4	4	8	4						
5	5	15	15	5					
6	6	24	38	24	6				
7	7	35	77	77	35	7			
8	8	48	136	192	136	48	8		
9	9	63	219	405	405	219	63	9	
10	10	80	330	760	1002	760	330	80	10

Table 4 Values of A_i in *s-Series*

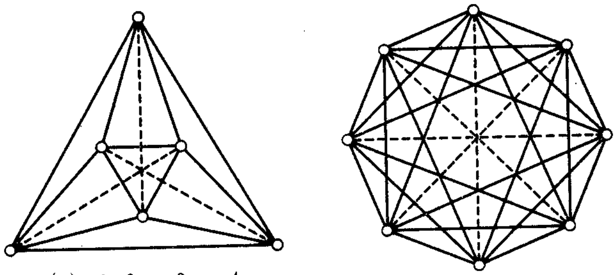
s	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}
1	1									
2	2	1								
3	3	4	1							
4	4	9	6	1						
5	5	16	19	8	1					
6	6	25	44	33	10	1				
7	7	36	85	96	51	12	1			
8	8	49	146	225	180	73	14	1		
9	9	64	231	456	501	304	99	16	1	
10	10	81	344	833	1182	985	476	129	18	1

6. Examples

A graph is called *regular of degree* ρ if all local degrees have the same value ρ. Some of regular graphs belong to r- or *m-Series* described in the previous chapter. For example, the regular graphs shown in Figs. 8 and 9 with their local degrees ρ, belong to r- and *m-Series*, respectively. In these graphs, the branches to be removed from the original complete graph are shown by dotted lines.

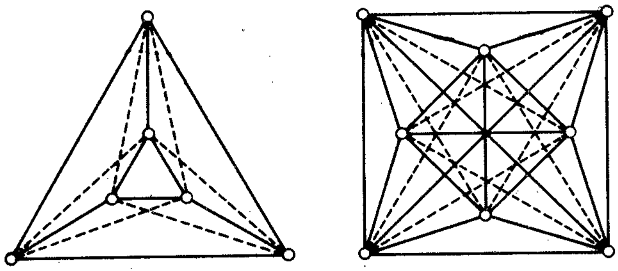
From Formula (9), the numbers of Hamiltonian circuits in the two regular graphs of Fig. 8 become (a) 16 and (b) 744, respectively, and from Formula (18), those in the two regular graphs of Fig. 9 become (a) 3 and (b) 177, respectively.

Fig. 10 shows three forms of the branches to be removed from a complete graph. The incomplete graphs made by removing these branches do not belong to six series described in the previous chapter. The number of Hamiltonian circuits in such the incomplete



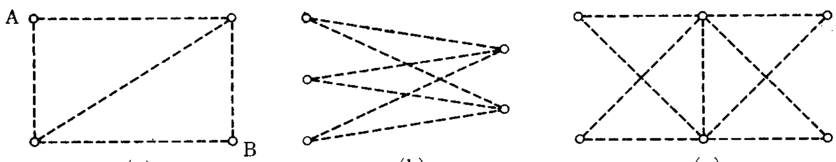
(a) $N=6, r=3, \rho=4$ (b) $N=8, r=4, \rho=6$

Fig. 8 Regular graphs belonging to *r-Series.*



(a) $N=6, m=6, \rho=3$ (b) $N=8, m=8, \rho=5$

Fig. 9 Regular graphs belonging to *m-Series.*



(a) (b) (c)

Fig. 10 Three forms of branches to be removed from a complete graph.

Table 5 Number of combinations of different branches

Removed graphs \ Patterns	$k=1$	$k=2$		$k=3$		$k=4$	
	(1)	(1)	(2)	(2)	(3)	(3)	(4)
Fig. 10 (a)	5	2	8	0	6	0	0
Fig. 10 (b)	6	6	9	6	12	0	6
Fig. 10 (c)	9	12	24	24	36	6	24

graphs can also be calculated by using the method described in Chapetr 5. Table 5 shows the number of combinations for making each of the patterns, shown in Fig. 4, by some of the branches shown in Fig. 10. Thus, the numbers of Hamiltonian circuits in incomplete graphs made by removing each of the three graphs, shown in Fig. 10, from a complete graph of N nodes are given by

$$\left.\begin{aligned} H_{(a)} &= (N-4)!\cdot(N-5)^2(N-6)/2, \\ H_{(b)} &= (N-5)!\cdot(N-5)(N-6)(N^2-11N+34)/2, \\ H_{(c)} &= (N-5)!\cdot(N-6)(N-7)^2(N-8)/2. \end{aligned}\right\} \tag{33}$$

As is evident from Eq. (33), these incomplete graphs have no Hamlitonian circuits when N equals to the number of nodes which are contained within the removed graphs. When $N=7$, we can find from the first equation in Eq. (33) that the incomplete graph derived by removing the graph of Fig. 10(a) from a complete graph has 12 Hamiltonian circuits. However, when another branch between nodes A and B in Fig. 10(a) is removed further from this incomplete graph, the remaining incomplete graph belongs to *c-Series* and as is evident from Formula (15), the Hamiltonian circuits are no longer in existence.

Fig. 11 shows two other forms of branches to be removed from a complete graph, and Table 6 gives the number of combinations for making each of the patterns, shown in Fig. 4, by some of the branches shown in Fig. 11. Thus, the numbers of Hamiltonian circuits in two incomplete graphs of N nodes under consideration are given by the following equations:

$$\begin{aligned} H_{(a)} &= (N-1)!/2-12\cdot(N-2)!+99\cdot(N-3)!-350\cdot(N-4)! \\ &\quad +540\cdot(N-5)!-330\cdot(N-6)!+60\cdot(N-7)! \\ &=(N-7)!\cdot(N-8)(N^5-37N^4+557N^3-4263N^2+16582N-26220)/2, \end{aligned} \tag{34}$$

$$\begin{aligned} H_{(b)} &= (N-1)!/2-12\cdot(N-2)!+108\cdot(N-3)!-464\cdot(N-4)!+1008\cdot(N-5)! \\ &\quad -1080\cdot(N-6)!+504\cdot(N-7)!-72\cdot(N-8)!. \end{aligned} \tag{35}$$

Since the value of $H_{(a)}$ becomes zero when $N=8$, it is obvious that the incomplete graph derived by removing the graph, shown in Fig. 11(a), from a complete graph has no Hamiltoian circuits when $N=7$. On the other hand, the value of $H_{(b)}$ does not become zero when $N=8$. In such a case, some of the branches to be removed from a complete

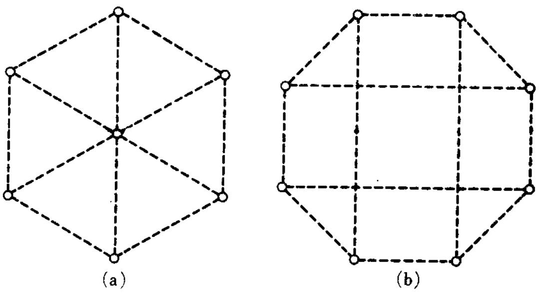


Fig. 11 Two forms of branches to be removed from a complete graph.

Table 6 Number of combinations of different branches

Patterns / Removed graphs	$k=1$	$k=2$		$k=3$			$k=4$				$k=5$			$k=6$		$k=7$
	(1)	(1)	(2)	(1)	(2)	(3)	(1)	(2)	(3)	(4)	(3)	(4)	(5)	(5)	(6)	(7)
Fig. 11 (a)	12	33	33	20	102	66	0	33	159	90	0	120	90	0	60	0
Fig. 11 (b)	12	42	24	44	120	48	9	96	240	72	108	264	120	204	96	72

graph form Hamiltonian circuits in the original complete graph. Consequently, in order to obtain the value of $H_{(b)}$ for $N=8$, we must add the number of Hamiltonian circuits in the graph of Fig. 11(b) to the value of Eq. (35) for $N=8$, as the term of δ_{mN} was added in *m-Series*. The number of Hamiltonian circuits in the graph of Fig. 11(b) is obtained easily by observing that graph and becomes 6. Thus, the number of Hamiltonian circuits in incomplete graph derived by removing the graph of Fig. 11(b) from a complete graph of $N=8$ becomes as follows:

$$H_{(b)}=7!/2-12\cdot 6!+108\cdot 5!-464\cdot 4!+1008\cdot 3!-1080\cdot 2!+504\cdot 1!-72\cdot 0!+6=30.$$

It goes without saying that the values of $H_{(b)}$ for $N\geqq 9$ can be obtained from Eq. (35).

7. Conclusions

In the above, we have investigated a method for finding the number of Hamiltonian circuits in incomplete graphs and have given the formulas for the six basic series of incomplete graphs.

The idea of "inversion of branches" was used as a concept for finding the number of Hamiltonian circuits and it was shown that this concept can also be applied to the incomplete graphs which do not belong to the standardized series. As is evident from Eq. (32), the number of Hamiltonian circuits in a given incomplete graph can be found by obtaining the values of coefficients A_i, and these coefficients are calculated from the number of combinations for making each of the patterns, shown in Fig. 4, by some of the branches to be removed from a complete graph to make that incomplete graph. If the complementary graph has a systematic form as the basic series described in Chapter 5 or has a comparatively simple form as the examples shown in the previous chapter, it is not difficult particularly to obtain the number of combinations just mentions above. However, when the complementary graph has a complex form and the number of branches in it much more than that in the given graph, calculating the number of such the combinations becomes a work that takes plenty of time.

Furthermore, when the number of nodes belonging to the complementary graph equals to that of the original complete graph, another considerable problem occurs. It is said that we must find the number of Hamiltonian circuits which are formed by some of the branches belonging to the complementary graph. The incomplete graphs belonging to *m-Series*, described in Chapter 5, and the incomplete graph, shown in the last of Chapter 6, are examples of such a case. These examples suggest us a contradiction that we must obtain the number of Hamiltonian circuits in the complementary graph for the purpose of obtaining the number of Hamiltonian circuits in a given incomplete graph. When the complementary graph has a systematic form or a simple form, the number of Hamiltonian circuits in that graph can be obtained easily by observation. However, when the form of complementary graph is complex, in general, there is no alternative but to obtain the number of Hamiltonian circuits in that graph by trial and error. This matter is no longer theoretically and does not agree with the purpose of this paper. Consequently, the establishment of simple method, which can also be applied to such a case, is expected.

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