Surface Displacements of the Semi-Infinite Solid Subjected to Paraboloidal Load Distributed over an Elliptical Area

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We consider the surface displacements of the semi-infinite body by using the general solutions given by Boussinesq. Within the loaded elliptical area horizontal displacements are readily obtained, while over the unloaded surface their values are calculated through the somewhat complicated procedures on the basis of the theory of residue.

The vertical displacements are given everywhere over the surface in the form of elliptic integrals.

The horizontal displacements direct toward the center of the loaded region at every point on the surface and have the maximum values at the inner points near the boundary, while the vertical displacements become maximum in the center of the elliptical area.

1. Introduction

In the theory of elasticity the boundary value problems are objects to which much research has been devoted. At an earlier date Boussinesq completed the theoretical outline, giving the solutions for a more general type of boundary conditions, since then various practical problems have been dealt with on the basis of his solutions.

In this paper we deal with the surface displacements of the semi-infinite solid under the distributed load in the elliptical area by using Love's solution given in the integral expressions. In practice the loaded area is considered to be elliptical, the assumption of which seems to be more general than the case of the uniformly loaded circular area which was solved by Timoshenko[1].

In the case of rolling, pressing, etc., the results of this problem seem to offer an effective suggestion of the behavior of the initial surface flow of metals to be compressed.

2. Notation

In this paper the following symbols are used:

(x, y, z)	coordinate of a point in the semi-infinite solid
$(\xi, \eta, 0)$	coordinate of a loaded point on the surface of the body
r	distance of a point $(x,y,0)$ from a point $(\xi,\eta,0)$
θ	angle between generator and r

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r_1, r_2	distances of a point $(x,y,0)$ from points on the elliptical boundary
θ_1, θ_2	angles between generator and two tangents to ellipse
$2a$, $2b$	major and minor axis of ellipse
e	eccentricity of ellipse
p	pressure acting on the surface of the body
u, v, w	components of displacement
z	complex variable $(=x'+iy')$
$\bar{z}$	conjugate of z $(=x'-iy')$
i	$=\sqrt{-1}$
$\boldsymbol{Res}(\phi(z_k))$	residue of function $\phi(z)$ at a point z_k
$\boldsymbol{Imag}\ F(z)$	imaginary part of function $F(z)$
λ, μ	Lamé's constants
$\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}, \boldsymbol{F}_1, \boldsymbol{G}_1, \boldsymbol{H}_1$ $\psi, \psi_1, \Omega, \chi$	harmonic functions

3. Analysis

3. 1. Fundamental equations

The solutions of this kind problem are given by Boussinesq. When the semi-infinite solid bounded by plane is subjected to the surface tractions, the expressions for displacements are given by Love[2] as follows:

$$\left.\begin{aligned} u &= \frac{1}{2\pi\mu}\frac{\partial \boldsymbol{F}}{\partial z}-\frac{1}{4\pi\mu}\frac{\partial \boldsymbol{H}}{\partial x}+\frac{\lambda}{4\pi\mu(\lambda+\mu)}\frac{\partial \psi_1}{\partial z}-\frac{1}{4\pi\mu}z\frac{\partial \psi}{\partial x} \\ v &= \frac{1}{2\pi\mu}\frac{\partial \boldsymbol{G}}{\partial z}-\frac{1}{4\pi\mu}\frac{\partial \boldsymbol{H}}{\partial y}+\frac{\lambda}{4\pi\mu(\lambda+\mu)}\frac{\partial \psi_1}{\partial y}-\frac{1}{4\pi\mu}z\frac{\partial \psi}{\partial y} \\ w &= \frac{1}{4\pi\mu}\frac{\partial \boldsymbol{H}}{\partial z}+\frac{1}{4\pi(\lambda+\mu)}\psi-\frac{1}{4\pi\mu}z\frac{\partial \psi}{\partial z} \end{aligned}\right\} \qquad (1)$$

where

$$\boldsymbol{F}=\iint X_\nu \chi d\xi d\eta\ ,\quad \boldsymbol{G}=\iint Y_\nu \chi d\xi d\eta\ ,\quad \boldsymbol{H}=\iint Z_\nu \chi d\xi d\eta$$

$$\boldsymbol{F}_1=\iint X_\nu \Omega d\xi d\eta\ ,\quad \boldsymbol{G}_1=\iint Y_\nu \Omega d\xi d\eta\ ,\quad \boldsymbol{H}_1=\iint Z_\nu \Omega d\xi d\eta$$

$$\psi=\frac{\partial \boldsymbol{F}}{\partial x}+\frac{\partial \boldsymbol{G}}{\partial y}+\frac{\partial \boldsymbol{H}}{\partial z}\ ,\quad \psi_1=\frac{\partial \boldsymbol{F}_1}{\partial x}+\frac{\partial \boldsymbol{G}_1}{\partial y}+\frac{\partial \boldsymbol{H}_1}{\partial z}$$

$$\chi=\log(z+r)\ ,\quad \Omega=z\log(z+r)$$

X_ν, Y_ν and Z_ν are surface tractions acting in the x–, y– and z–directions respectively, and the double integrations are taken over the elliptical area subjected to pressure.

Eqs. (1) represent displacements at any point (x,y,z) in the body produced by surface tractions acting at the point $(\xi,\eta,0)$ on the surface of the body.

3.2 Assumptions

For solving this problem, we take the following two assumptions:

i) The area subjected to pressure is the region corresponding to the following relation

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}\leqq 1 \tag{2}$$

ii) The pressure acts solely in the direction of z-axis and is denoted by the following expression

$$p = p_0\left(1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}\right) \tag{3}$$

where p_0 is the maximum pressure in the center of ellipse.

3. 3 Calculation

With the above assumption ii) eqs. (1) are written as

$$\left.\begin{aligned}
u &= -\frac{1}{4\pi\mu}\frac{\partial \boldsymbol{H}}{\partial x}+\frac{\lambda}{4\pi\mu(\lambda+\mu)}\frac{\partial^2 \boldsymbol{H}_1}{\partial x\partial z}-\frac{1}{4\pi\mu}z\frac{\partial^2 \boldsymbol{H}}{\partial x\partial z}\\
v &= -\frac{1}{4\pi\mu}\frac{\partial \boldsymbol{H}}{\partial y}+\frac{\lambda}{4\pi\mu(\lambda+\mu)}\frac{\partial^2 \boldsymbol{H}_1}{\partial y\partial z}-\frac{1}{4\pi\mu}z\frac{\partial^2 \boldsymbol{H}}{\partial y\partial z}\\
w &= \frac{\lambda+2\eta}{4\pi\mu(\lambda+\mu)}\frac{\partial \boldsymbol{H}}{\partial z}-\frac{1}{4\pi\mu}z\frac{\partial^2 \boldsymbol{H}}{\partial z^2}
\end{aligned}\right\} \tag{4}$$

Substituting eq. (3) into eqs. (4), we obtain

$$\left.\begin{aligned}
u &= -\frac{1}{4\pi(\lambda+\mu)}\iint\frac{p(x-\xi)}{r(z+r)}d\xi d\eta+\frac{z}{4\pi\mu}\iint\frac{p(x-\xi)}{r^3}d\xi d\eta\\
v &= -\frac{1}{4\pi(\lambda+\mu)}\iint\frac{p(y-\eta)}{r(z+r)}d\xi d\eta+\frac{z}{4\pi\mu}\iint\frac{p(y-\eta)}{r^3}d\xi d\eta\\
w &= \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}\iint\frac{p}{r}d\xi d\eta+\frac{z^2}{4\pi\mu}\iint\frac{p}{r^3}d\xi d\eta
\end{aligned}\right\} \tag{5}$$

Substituting $z=0$ into eqs. (5), finally the surface displacements of the semi-infinite solid are given as follows;

$$\left.\begin{aligned}
u &= -\frac{1}{4\pi(\lambda+\mu)}\iint\frac{p(x-\xi)}{r}d\xi d\eta\\
v &= -\frac{1}{4\pi(\lambda+\mu)}\iint\frac{p(y-\eta)}{r^2}d\xi d\eta\\
w &= \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)}\iint\frac{p}{r}d\xi d\eta
\end{aligned}\right\} \tag{6}$$

where the double integrations are taken over the elliptical area.

For the calculation of eqs. (6) we transform Cartesian coordinates $(\xi,\ \eta)$ to polar coordinates (r, ξ). From Fig. 1 we obtain

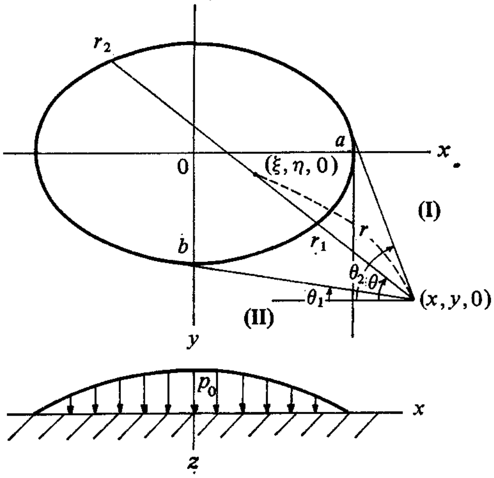

Fig. 1. A model of the region under the paraboloidal load.

$$x-\xi = r\cos\theta\,, \qquad y-\eta = r\sin\theta\,.$$

Using this relation, eq. (3) is written as

$$p = p_0(A+2Br-Cr^2) \tag{3$'$}$$

where

$$A = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\,, \quad B = \frac{x\cos\theta}{a^2}+\frac{y\sin\theta}{b^2}\,, \quad C = \frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$$

Substituting these into eqs. (6), we obtain within the ellipse

$$\left.\begin{aligned} u &= -\frac{p_0}{4\pi(\lambda+\mu)}\int_0^{r_2}\int_0^{2\pi}(A+2Br-Cr^2)\cos\theta\,drd\theta = -\frac{p_0}{4\pi(\lambda+\mu)}\int_0^{2\pi}\boldsymbol{G}(\theta)\cos\theta d\theta \\ v &= -\frac{p_0}{4\pi(\lambda+\mu)}\int_0^{r_2}\int_0^{2\pi}(A+2Br-Cr^2)\sin\theta\,drd\theta = -\frac{p_0}{4\pi(\lambda+\mu)}\int_0^{2\pi}\boldsymbol{G}(\theta)\sin\theta d\theta \\ w &= \frac{(\lambda+2\mu)p_0}{4\pi\mu(\lambda+\mu)}\int_0^{r_2}\int_0^{2\pi}(A+2Br-Cr^2)drd\theta = \frac{(\lambda+2\mu)p_0}{4\pi\mu(\lambda+\mu)}\int_0^{2\pi}\boldsymbol{G}(\theta)d\theta \end{aligned}\right\} \tag{7}$$

and without the ellipse

$$\left.\begin{aligned} u &= -\frac{p_0}{4\pi(\lambda+\mu)}\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}(A+2Br-Cr^2)\cos\theta drd\theta = -\frac{p_0}{4\pi(\lambda+\mu)}\int_{\theta_1}^{\theta_2}\boldsymbol{H}(\theta)\cos\theta d\theta \\ v &= -\frac{p_0}{4\pi(\lambda+\mu)}\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}(A+2Br-Cr^2)\sin\theta drd\theta = -\frac{p_0}{4\pi(\lambda+\mu)}\int_{\theta_1}^{\theta_2}\boldsymbol{H}(\theta)\sin\theta d\theta \\ w &= \frac{(\lambda+2\mu)p_0}{4\pi\mu(\lambda+\mu)}\int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2}(A+2Br-Cr^2)drd\theta = \frac{(\lambda+2\mu)p_0}{4\pi\mu(\lambda+\mu)}\int_{\theta_1}^{\theta_2}\boldsymbol{H}(\theta)d\theta \end{aligned}\right\} \tag{8}$$

where

$$G(\theta)=Ar_2+Br_2^2-\frac{C}{3}r_2^3\,,\quad H(\theta)=A(r_2-r_1)+B(r_2^2-r_1^2)-\frac{C}{3}(r_2^3-r_1^3)$$

and the following expressions are used

$$r_1=\frac{b^2x\cos\theta+a^2y\sin\theta-ab\sqrt{a^2\sin^2\theta+b^2\cos^2\theta-(x\sin\theta-y\cos\theta)^2}}{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$r_2=\frac{b^2x\cos\theta+a^2y\sin\theta+ab\sqrt{a^2\sin^2\theta+b^2\cos^2\theta-(x\sin\theta-y\cos\theta)^2}}{a^2\sin^2\theta+b^2\cos^2\theta}$$

and angles θ_1 and θ_2 are given as follows;

$$\theta_1=\tan^{-1}\frac{xy-\sqrt{b^2x^2+a^2y^2-a^2b^2}}{x^2-a^2}$$

$$\theta_2=\tan^{-1}\frac{xy+\sqrt{b^2x^2+a^2y^2-a^2b^2}}{x^2-a^2}$$

where

$$x^2-a^2\neq 0\,.$$

3.3.1 Horizontal displacements at a point $(x, y, 0)$ within the elliptical area

After integration of eqs. (7), we obtain

$$\left.\begin{aligned} u&=\frac{p_0}{2(\lambda+\mu)}\left\{\frac{b(b+2a)}{3a^2(a+b)^2}x^3+\frac{xy^2}{(a+b)^2}-\frac{bx}{a+b}\right\}\\ v&=\frac{p_0}{2(\lambda+\mu)}\left\{\frac{a(a+2b)}{3b^2(a+b)^2}y^3+\frac{x^2y}{(a+b)^2}-\frac{ay}{a+b}\right\}\end{aligned}\right\}\qquad(9)$$

3.3.2 Horizontal displacements at a point $(x, y, 0)$ without the elliptical area

i) For a point $(x, y, 0)$ in a region (I) $\quad(|x|>a)$

Eqs. (8) can be written in the form

$$\left.\begin{aligned} u&=-\frac{p_0}{2\pi(\lambda+\mu)}\left(\frac{A}{3}I_{u1}+\frac{1}{6}I_{u2}\right)\\ v&=-\frac{p_0}{2\pi(\lambda+\mu)}\left(\frac{A}{3}I_{v1}+\frac{1}{6}I_{v2}\right)\end{aligned}\right\}\qquad(10)$$

where

$$\left.\begin{aligned} I_{u1}&=2ab\int_{\theta_1}^{\theta_2}\frac{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta-(x\sin\theta-y\cos\theta)^2}}{a^2\sin^2\theta+b^2\cos^2\theta}\cos\theta\,d\theta\\ I_{u2}&=\frac{4}{ab}\int_{\theta_1}^{\theta_2}\frac{(b^2x\cos\theta+a^2y\sin\theta)^2\sqrt{a^2\sin^2\theta+b^2\cos^2\theta-(x\sin\theta-y\cos\theta)^2}}{(a^2\sin^2\theta+b^2\cos^2\theta)^2}\cos\theta\,d\theta\\ I_{v1}&=2ab\int_{\theta_1}^{\theta_2}\frac{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta-(x\sin\theta-y\cos\theta)^2}}{a^2\sin^2\theta+b^2\cos^2\theta}\sin\theta\,d\theta\\ I_{v2}&=\frac{4}{ab}\int_{\theta_1}^{\theta_2}\frac{(b^2x\cos\theta+a^2y\sin\theta)^2\sqrt{a^2\sin^2\theta+b^2\cos^2\theta-(x\sin\theta-y\cos\theta)^2}}{(a^2\sin^2\theta+b^2\cos^2\theta)^2}\sin\theta\,d\theta\end{aligned}\right\}\qquad(11)$$

Putting $\tan\theta = t$, eqs. (11) are expressed as follows;

$$\left.\begin{aligned}
I_{u1} &= 2ab\int_{t_1}^{t_2}\frac{\sqrt{(a^2-x^2)t^2+2xyt+b^2-y^2}}{(a^2t^2+b^2)(t^2+1)}\,dt \\
I_{u2} &= \frac{4}{ab}\int_{t_1}^{t_2}\frac{(a^2yt+b^2x)^2\sqrt{(a^2-x^2)t^2+2xyt+b^2-y^2}}{(a^2t^2+b^2)^2(t^2+1)}\,dt \\
I_{v1} &= 2ab\int_{t_1}^{t_2}\frac{\sqrt{(a^2-x^2)t^2+2xyt+b^2-y^2}}{(a^2t^2+b^2)(t^2+1)}\,tdt \\
I_{v2} &= \frac{4}{ab}\int_{t_1}^{t_2}\frac{(a^2yt+b^2x)^2\sqrt{(a^2-x^2)t^2+2xyt+b^2-y^2}}{(a^2t^2+b^2)^2(t^2+1)}\,tdt
\end{aligned}\right\} \tag{12}$$

where

$$\left.\begin{aligned}
t_1 &= \tan\theta_1 = \frac{xy-\sqrt{b^2x^2+a^2y^2-a^2b^2}}{x^2-a^2} \\
t_2 &= \tan\theta_2 = \frac{xy+\sqrt{b^2x^2+a^2y^2-a^2b^2}}{x^2-a^2}
\end{aligned}\right\} \tag{13}$$

Introducing the following new variable K

$$\sqrt{(a^2-x^2)t^2+2xyt+b^2-y^2} = (t_2-t)K \qquad (K>0)$$

eqs. (12) are written

$$\left.\begin{aligned}
I_{u1} &= 2ab(x^2-a^2)^2(t_2-t_1)^2 \\
&\times\int_{-\infty}^{\infty}\frac{K^2(K^2+x^2-a^2)dK}{[\{t_2K^2+t_1(x^2-a^2)\}^2+(K^2+x^2-a^2)^2][a^2\{t_2K^2+t_1(x^2-a^2)\}^2+b^2(K^2+x^2-a^2)^2]} \\
I_{u2} &= \frac{4}{ab}(x^2-a^2)^2(t_2-t_1)^2 \\
&\times\int_{-\infty}^{\infty}\frac{\{(a^2t_2y+b^2x)K^2+(a^2t_1y+b^2x)(x^2-a^2)\}^2K^2(K^2+x^2-a^2)dK}{[\{t_2K^2+t_1(x^2-a^2)\}^2+(K^2+x^2-a^2)^2][a^2\{t_2K^2+t_1(x^2-a^2)\}^2+b^2(K^2+x^2-a^2)^2]^2} \\
I_{v1} &= 2ab(x^2-a^2)^2(t_2-t_1)^2 \\
&\times\int_{-\infty}^{\infty}\frac{K^2\{t_2K^2+t_1(x^2-a^2)\}dK}{[\{t_2K^2+t_1(x^2-a^2)\}^2+(K^2+x^2-a^2)^2][a^2\{t_2K^2+t_1(x^2-a^2)\}^2+b^2(K^2+x^2-a^2)^2]} \\
I_{v2} &= \frac{4}{ab}(x^2-a^2)^2(t_2-t_1)^2 \\
&\times\int_{-\infty}^{\infty}\frac{\{(a^2t_2y+b^2x)K^2+(a^2t_1y+b^2x)(x^2-a^2)\}^2K^2\{t_2K^2+t_1(x^2-a^2)\}dK}{[\{t_2K^2+t_1(x^2-a^2)\}^2+(K^2+x^2-a^2)^2][a^2\{t_2K^2+t_1(x^2-a^2)\}^2+b^2(K^2+x^2-a^2)^2]^2}
\end{aligned}\right\} \tag{12'}$$

Again replacing the variable K by the complex variable z, we obtain

$$\left.\begin{aligned} \boldsymbol{I}_{u1} &= 2ab(x^2-a^2)^2(t_2-t_1)^2\int_{-\infty}^{\infty}\phi_{u1}(z)dz \\ \boldsymbol{I}_{u2} &= \frac{4}{ab}(x^2-a^2)^2(t_2-t_1)^2\int_{-\infty}^{\infty}\phi_{u2}(z)dz \\ \boldsymbol{I}_{v1} &= 2ab(x^2-a^2)^2(t_2-t_1)^2\int_{-\infty}^{\infty}\phi_{v1}(z)dz \\ \boldsymbol{I}_{v2} &= \frac{4}{ab}(x^2-a^2)^2(t_2-t_1)^2\int_{-\infty}^{\infty}\phi_{v2}(z)dz \end{aligned}\right\} \quad (12)''$$

where

$$\phi_{u1}(z) = \frac{z^2(z^2+x^2-a^2)}{[\{t_2z^2+t_1(x^2-a^2)\}^2+(z^2+x^2-a^2)^2][a^2\{t_2z^2+t_1(x^2-a^2)\}^2+b^2(z^2+x^2-a^2)^2]}$$

$$\phi_{u2}(z) = \frac{\{(a^2t_2y+b^2x)z^2+(a^2t_1y+b^2x)(x^2-a^2)\}^2z^2(z^2+x^2-a^2)}{[\{t_2z^2+t_1(x^2-a^2)\}^2+(z^2+x^2-a^2)^2][a^2\{t_2z^2+t_1(x^2-a^2)\}^2+b^2(z^2+x^2-a^2)^2]^2}$$

$$\phi_{v1}(z) = \frac{z^2\{t_2z^2+t_1(x^2-a^2)\}}{[\{t_2z^2+t_1(x^2-a^2)\}^2+(z^2+x^2-a^2)^2][a^2\{t_2z^2+t_1(x^2-a^2)\}^2+b^2(z^2+x^2-a^2)^2]}$$

$$\phi_{v2}(z) = \frac{\{(a^2t_2y+b^2x)z^2+(a^2t_1y+b^2x)(x^2-a^2)\}^2z^2\{t_2z^2+t_1(x^2-a^2)\}}{[\{t_2z^2+t_1(x^2-a^2)\}^2+(z^2+x^2-a^2)^2][a^2\{t_2z^2+t_1(x^2-a^2)\}^2+b^2(z^2+x^2-a^2)^2]^2}$$

For the calculation of eqs. (12)″, we use the residue theorem. The singular points of the integrand are obtained with the following equations.

$$a^2\{t_2z^2+t_1(x^2-a^2)\}^2+b^2(z^2+x^2-a^2)^2 = 0 \quad ①$$

$$\{t_2z^2+t_1(x^2-a^2)\}^2+(z^2+x^2-a^2)^2 = 0 \quad ②$$

Among the singular points obtained from eqs. ① and ②, those which exist in the upper half of z-plane are given as follows (Fig. 2).

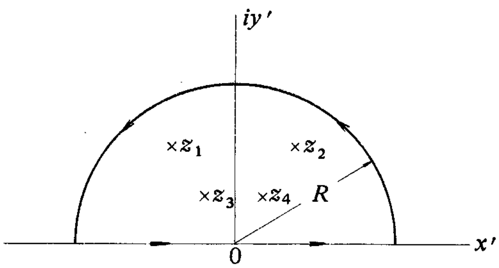

Fig. 2. Path of integration and positions of singular points.

From eq. ①

$$z_1 = -\sqrt{\frac{\sqrt{H_2}-H_1}{2}}+\sqrt{\frac{\sqrt{H_2}+H_1}{2}}\,i$$

$$z_2 = \sqrt{\frac{\sqrt{H_2}-H_1}{2}}+\sqrt{\frac{\sqrt{H_2}+H_1}{2}}\,i$$

From eq. ②

$$z_3 = -\sqrt{\frac{\sqrt{H_4}-H_3}{2}}+\sqrt{\frac{\sqrt{H_4}+H_3}{2}}\,i$$

$$z_4 = \sqrt{\frac{\sqrt{H_4}-H_3}{2}}+\sqrt{\frac{\sqrt{H_4}+H_3}{2}}\,i$$

where

$$H_1 = \frac{b^2x^2+a^2y^2-2a^2b^2}{a^2t_2^{\,2}+b^2}, \quad H_2 = \frac{(a^2t_1^{\,2}+b^2)(x^2-a^2)^2}{a^2t_2^{\,2}+b^2}$$

$$H_3 = \frac{x^2+y^2-a^2-b^2}{t_2^{\,2}+1}, \quad H_4 = \frac{(x^2-y^2-a^2+b^2)^2+4x^2y^2}{(t_2^{\,2}+1)^2}$$

For example, considering the integration $\int_{-\infty}^{\infty} \phi_{u1}\, dz$ along the semi-circle C_R, we have the following formula

$$\int_{-R}^{R} \phi_{u1}\, dz + \int_{C_R} \phi_{u1}\, dz = 2\pi i \sum_{k=1}^{4} (\boldsymbol{Res}(\phi_{u1}(z_k)))\,.$$

Approaching the radius R of semi-circle C_R to infinity, the integration along the semi-circle vanishes, and the above expression becomes

$$\int_{-\infty}^{\infty} \phi_{u1}\, dz = 2\pi i \sum_{k=1}^{4} \boldsymbol{Res}(\phi_{u1}(z_k))\,.$$

Using this formula, eqs. (12)″ are written as

$$\left.\begin{aligned}
\boldsymbol{I}_{u1} &= -16ab(b^2x^2+a^2y^2-a^2b^2)\pi\ \boldsymbol{Imag} \sum_{k=1}^{4} \boldsymbol{Res}(\phi_{u1}(z_k)) \\
\boldsymbol{I}_{u2} &= -\frac{32}{ab}(b^2x^2+a^2y^2-a^2b^2)\pi\ \boldsymbol{Imag} \sum_{k=1}^{4} \boldsymbol{Res}(\phi_{u2}(Z_k)) \\
\boldsymbol{I}_{v1} &= -16ab(b^2x^2+a^2y^2-a^2b^2)\pi\ \boldsymbol{Imag} \sum_{k=1}^{4} \boldsymbol{Res}(\phi_{v1}(Z_k)) \\
\boldsymbol{I}_{v2} &= -\frac{32}{ab}(b^2x^2+a^2y^2-a^2b^2)\pi\ \boldsymbol{Imag} \sum_{k=1}^{4} \boldsymbol{Res}(\phi_{v2}(Z_k))
\end{aligned}\right\} \quad (14)$$

where

$$\boldsymbol{Res}(\phi_{u1}(z_1)) = \frac{z_1}{2D(a^2t_2^{\,2}+b^2)(z_1^{\,2}-\bar{z}_1^{\,2})(z_1^{\,2}+R)}$$

$$\boldsymbol{Res}(\phi_{u1}(z_2)) = \frac{\bar{z}_1}{2D(a^2t_2^{\,2}+b^2)(z_1^{\,2}-\bar{z}_1^{\,2})(\bar{z}_1^{\,2}+R)}$$

$$\boldsymbol{Res}(\phi_{u1}(z_3)) = -\frac{z_3}{2(t_2^{\,2}+1)(a^2-b^2)(z_3^{\,2}-\bar{z}_3^{\,2})(z_3^{\,2}+R)}$$

$$\boldsymbol{Res}(\phi_{u1}(z_4)) = -\frac{\bar{z}_3}{2(t_2^{\,2}+1)(a^2-b^2)(z_3^{\,2}-\bar{z}_3^{\,2})(\bar{z}_3^{\,2}+R)}$$

$$\boldsymbol{Res}(\phi_{u2}(z_1)) = \frac{(\alpha_1 z_1{}^2+\alpha_2 R)}{4(a^2t_2{}^2+b^2)^2D^2 z_1(z_1{}^2-\bar{z}_1{}^2)^3(z_1{}^2+R)^3}$$
$$\times[2D(z_1{}^2-\bar{z}_1{}^2)(z_1{}^2+R)\{(2z_1{}^2+R)(\alpha_1 z_1{}^2+\alpha_2 R)+2z_1{}^2(z_1{}^2+R)\alpha_1\}$$
$$-(\alpha_1 z_1{}^2+\alpha_2 R)\{D(5z_1{}^2-\bar{z}_1{}^2)(z_1{}^2+R)^2+4z_1{}^2(z_1{}^2-\bar{z}_1{}^2)$$
$$\times\left(t_2(t_2 z_1{}^2+t_1 R)+z_1{}^2+R\right)\}]$$

$$\boldsymbol{Res}(\phi_{u2}(z_2)) = \frac{-(\alpha_1\bar{z}_1{}^2+\alpha_2 R)}{4(a^2t_2{}^2+b^2)^2D^2\bar{z}_1(z_1{}^2-\bar{z}_1{}^2)^3(\bar{z}_1{}^2+R)^3}$$
$$\times[2D(z_1{}^2-\bar{z}_1{}^2)(\bar{z}_1{}^2+R)\{(2\bar{z}_1{}^2+R)(\alpha_1\bar{z}_1{}^2+\alpha_2 R)$$
$$+2\bar{z}_1{}^2(\bar{z}_1{}^2+R)\alpha_1\}+(\alpha_1\bar{z}_1{}^2+\alpha_2 R)\{D(5\bar{z}_1{}^2-z_1{}^2)(\bar{z}_1{}^2+R)^2$$
$$-4\bar{z}_1{}^2(z_1{}^2-\bar{z}_1{}^2)\left(t_2(t_2\bar{z}_1{}^2+t_1 R)+\bar{z}_1{}^2+R\right)\}]$$

$$\boldsymbol{Res}(\phi_{u2}(z_3)) = \frac{z_3(\alpha_1 z_3{}^2+\alpha_2 R)^2}{2(t_2{}^2+1)(a^2-b^2)^2(z_3{}^2-\bar{z}_3{}^2)(z_3{}^2+R)^3}$$

$$\boldsymbol{Res}(\phi_{u2}(z_4)) = \frac{\bar{z}_3(\alpha_1\bar{z}_3{}^2+\alpha_2 R)^2}{2(t_2{}^2+1)(a^2-b^2)^2(z_3{}^2-\bar{z}_3{}^2)(\bar{z}_3{}^2+R)^3}$$

$$\boldsymbol{Res}(\phi_{v1}(z_1)) = \frac{z_1(t_2 z_1{}^2+t_1 R)}{2(a^2t_2{}^2+b^2)D(z^2{}_1-\bar{z}_1{}^2)(z_1{}^2+R)^2}$$

$$\boldsymbol{Res}(\phi_{v1}(z_2)) = \frac{\bar{z}_1(t_2\bar{z}_1{}^2+t_1 R)}{2(a^2t_2{}^2+b^2)D(z_1{}^2-\bar{z}_1{}^2)(\bar{z}_1{}^2+R)^2}$$

$$\boldsymbol{Res}(\phi_{v1}(z_3)) = -\frac{z_3(t_2 z_3{}^2+t_1 R)}{2(t_2{}^2+1)(a^2-b^2)(z_3{}^2-\bar{z}_3{}^2)(z_3{}^2+R)^2}$$

$$\boldsymbol{Res}(\phi_{v1}(z_4)) = -\frac{\bar{z}_3(t_2\bar{z}_3{}^2+t_1 R)}{2(t_2{}^2+1)(a^2-b^2)(z_3{}^2-\bar{z}_3{}^2)(\bar{z}_3{}^2+R)^2}$$

$$\boldsymbol{Res}(\phi_{v2}(z_1)) = \frac{\alpha_1 z_1{}^2+\alpha_2 R}{4(a^2t_2{}^2+b^2)^2D^2 z_1(z_1{}^2-\bar{z}_1{}^2)^3(z_1{}^2+R)^4}$$
$$\times[2D(z_1{}^2-\bar{z}_1{}^2)(z_1{}^2+R)^2\{2\alpha_1 z_1{}^2(t_2 z_1{}^2+t_1 R)$$
$$+(2t_2 z_1{}^2+t_1 R)(\alpha_1 z_1{}^2+\alpha_2 R)\}-(t_2 z_1{}^2+t_1 R)(\alpha_1 z_1{}^2+\alpha_2 R)$$
$$\times\{D(5z_1{}^2-\bar{z}_1{}^2)(z_1{}^2+R)^2+4z_1{}^2(z_1{}^2-\bar{z}_1{}^2)\left(t_2(t_2 z_1{}^2+t_1 R)+z_1{}^2+R\right)\}]$$

$$\boldsymbol{Res}(\phi_{v2}(z_2)) = \frac{-(\alpha_1\bar{z}_1{}^2+\alpha_2 R)}{4(a^2t_2{}^2+b^2)^2D^2\bar{z}_1(z_1{}^2-\bar{z}_1{}^2)^3(\bar{z}_1{}^2+R)^4}$$
$$\times[2D(z_1{}^2-\bar{z}_1{}^2)(\bar{z}_1{}^2+R)^2\{2\alpha_1\bar{z}_1{}^2(t_2\bar{z}_1{}^2+t_1 R)$$
$$+(2t_2\bar{z}_1{}^2+t_1 R)(\alpha_1\bar{z}_1{}^2+\alpha_2 R)\}+(t_2\bar{z}_1{}^2+t_1 R)(\alpha_1\bar{z}_1{}^2+\alpha_2 R)$$
$$\times\{D(5\bar{z}_1{}^2-z_1{}^2)(\bar{z}_1{}^2+R)^2-4\bar{z}_1{}^2(z_1{}^2-\bar{z}_1{}^2)\left(t_2(t_2\bar{z}_1{}^2+t_1 R)+\bar{z}_1{}^2+R\right)\}]$$

$$\boldsymbol{Res}(\phi_{v2}(z_3)) = \frac{z_3(t_2 z_3{}^2+t_1 R)(\alpha_1 z_3{}^2+\alpha_2 R)^2}{2(t_2{}^2+1)(a^2-b^2)^2(z_3{}^2-\bar{z}_3{}^2)(z_3{}^2+R)^4}$$

$$\boldsymbol{Res}(\phi_{v2}(z_4)) = \frac{\bar{z}_3(t_2\bar{z}_3{}^2+t_1 R)(\alpha_1\bar{z}_3{}^2+\alpha_2 R)^2}{2(t_2{}^2+1)(a^2-b^2)^2(z_3{}^2-\bar{z}_3{}^2)(\bar{z}_3{}^2+R)^4}$$

$$\because\quad D=\frac{a^2-b^2}{a^2},\quad R=x^2-a^2,\quad \alpha_1=a^2t_2 y+b^2x,\quad \alpha_2=a^2t_1 y+b^2x$$

Substituting these expressions into the following equations, displacements u and v are obtained as

$$\left.\begin{aligned} u &= \frac{8p_0(b^2x^2+a^2y^2-a^2b^2)}{3(\lambda+\mu)}\left\{abA\ \boldsymbol{Imag}\sum_{k=1}^{4}\boldsymbol{Res}(\phi_{u1}(z_k))+\frac{1}{ab}\ \boldsymbol{Imag}\sum_{k=1}^{4}\boldsymbol{Res}(\phi_{u2}(z_k))\right\} \\ v &= \frac{8p_0(b^2x^2+a^2y^2-a^2b^2)}{3(\lambda+\mu)}\left\{abA\ \boldsymbol{Imag}\sum_{k=1}^{4}\boldsymbol{Res}(\phi_{v1}(z_k))+\frac{1}{ab}\ \boldsymbol{Imag}\sum_{k=1}^{4}\boldsymbol{Res}(\phi_{v2}(z_k))\right\} \end{aligned}\right\} \tag{15}$$

ii) For a point $(x, y, 0)$ in a region (II) $(0\leqq|x|<a)$

We introduce a new variable φ by using the following transformation

$$\varphi = \theta-\varphi'$$

in which

$$\varphi' = \frac{\theta_1+\theta_2}{2}$$

θ_1 and θ_2 are calculated in eqs. (13).

Through this transformation, eqs. (11) are written as

$$\left.\begin{aligned} \boldsymbol{I}_{u1} &= 2ab\int_{\varphi_1}^{\varphi_2}\sqrt{\delta\sin^2\varphi+\nu\cos^2\varphi}\,(-\sin\varphi'\cos\varphi+\cos\varphi'\sin\varphi) \\ &\times\frac{d\varphi}{(a^2\cos^2\varphi'+b^2\sin^2\varphi')\sin^2\varphi+2(a^2-b^2)\sin\varphi'\cos\varphi'\sin\varphi\cos\varphi+(a^2\sin^2\varphi'+b^2\cos^2\varphi')\cos^2\varphi} \\ \boldsymbol{I}_{u2} &= \frac{4}{ab}\int_{\varphi_1}^{\varphi_2}\{(b^2x\sin\varphi'-a^2y\cos\varphi')\sin\varphi-(b^2x\cos\varphi'+a^2y\sin\varphi')\cos\varphi\}^2\sqrt{\delta\sin^2\varphi+\nu\cos^2\varphi} \\ &\times\frac{(-\sin\varphi'\sin\varphi+\cos\varphi'\cos\varphi)d\varphi}{\{(a^2\cos^2\varphi'+b^2\sin^2\varphi')\sin^2\varphi+2(a^2-b^2)\sin\varphi'\cos\varphi'\sin\varphi\cos\varphi+(a^2\sin^2\varphi'+b^2\cos^2\varphi')\cos^2\varphi\}^2} \\ \boldsymbol{I}_{v1} &= 2ab\int_{\varphi_1}^{\varphi_2}\sqrt{\delta\sin^2\varphi+\nu\cos^2\varphi}\,(\cos\varphi'\sin\varphi+\sin\varphi'\cos\varphi) \\ &\times\frac{d\varphi}{(a^2\cos^2\varphi'+b^2\sin^2\varphi')\sin^2\varphi+2(a^2-b^2)\sin\varphi'\cos\varphi'\sin\varphi\cos\varphi+(a^2\sin^2\varphi'+b^2\cos^2\varphi')\cos^2\varphi} \\ \boldsymbol{I}_{v2} &= \frac{4}{ab}\int_{\varphi_1}^{\varphi_2}\{(b^2x\sin\varphi'-a^2y\cos\varphi')\sin\varphi-(b^2x\cos\varphi'+a^2y\sin\varphi')\cos\varphi\}^2\sqrt{\delta\sin^2\varphi+\nu\cos^2\varphi} \\ &\times\frac{(\cos\varphi'\sin\varphi+\sin\varphi'\cos\varphi)d\varphi}{\{a^2\cos^2\varphi'+b^2\sin^2\varphi')\sin^2\varphi+2(a^2-b^2)\sin\varphi'\cos\varphi'\sin\varphi\cos\varphi+(a^2\sin^2\varphi'+b^2\cos^2\varphi')\cos^2\varphi\}^2} \end{aligned}\right\} \tag{16}$$

where

$$\varphi_1 = -\varphi_2 = \frac{\theta_1-\theta_2}{2}$$

$$\delta = a^2\cos^2\varphi'+b^2\sin^2\varphi'-(x\cos\varphi'+y\sin\varphi')^2$$

$$\nu = a^2\sin^2\varphi'+b^2\cos^2\varphi'-(x\sin\varphi'+y\cos\varphi')^2$$

Putting $\tan\varphi = t$, then eqs. (16) are expressed as follows;

$$\left.\begin{aligned}
I_{u1} &= 2ab\int_{t_1'}^{t_2'} \frac{\sqrt{\delta t^2+\nu}\,(-t\sin\varphi'+\cos\varphi')}{(a^2\cos^2\varphi'+b^2\sin^2\varphi')t^2+2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t+a^2\sin^2\varphi'+b^2\cos^2\varphi'}\cdot\frac{dt}{1+t^2}\\
I_{u2} &= \frac{4}{ab}\int_{t_1'}^{t_2'}\{(b^2x\sin\varphi'-a^2y\cos\varphi')t-(b^2x\cos\varphi'+a^2y\sin\varphi')\}^2\\
&\quad\times\frac{\sqrt{\delta t^2+\nu}\,(-t\sin\varphi'+\cos\varphi')}{\{(a^2\cos^2\varphi'+b^2\sin^2\varphi')t^2+2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t+a^2\sin^2\varphi'+b^2\cos^2\varphi'\}^2}\cdot\frac{dt}{1+t^2}\\
I_{v1} &= 2ab\int_{t_1'}^{t_2'} \frac{\sqrt{\delta t^2+\nu}\,(t\cos\varphi'+\sin\varphi')}{(a^2\cos^2\varphi'+b^2\sin^2\varphi')t^2+2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t+a^2\sin^2\varphi'+b^2\cos^2\varphi'}\cdot\frac{dt}{1+t^2}\\
I_{v2} &= \frac{4}{ab}\int_{t_1'}^{t_2'}\{(b^2x\sin\varphi'-a^2y\cos\varphi')t-(b^2x\cos\varphi'+a^2y\sin\varphi')\}^2\\
&\quad\times\frac{\sqrt{\delta t^2+\nu}\,(t\cos\varphi'+\sin\varphi')}{\{(a^2\cos^2\varphi'+b^2\sin^2\varphi')t^2+2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t+a^2\sin^2\varphi'+b^2\cos^2\varphi'\}^2}\cdot\frac{dt}{1+t^2}
\end{aligned}\right\} \tag{16'}$$

where

$$t_1' = \tan\varphi_1 = -\sqrt{-\frac{\nu}{\delta}}, \qquad t_2' = \tan\varphi_2 = \sqrt{-\frac{\nu}{\delta}}$$

Introducing a complex variabe z and calculating in the same way as mentioned above, displacements u and v are obtained as follows;

$$\left.\begin{aligned}
u &= \frac{8p_0}{3(\lambda+\mu)}\cdot\frac{\delta^2\nu}{(\nu-\delta)}\\
&\quad\times\Bigg[\frac{abA\ \boldsymbol{Imag}\sum_{k=1}^{4}\boldsymbol{Res}(\psi_{u1}(z_k))}{(a^2\cos^2\varphi'+b^2\sin^2\varphi')t_2'^2+2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t_2'+a^2\sin^2\varphi'+b^2\cos^2\varphi'}\\
&\quad+\frac{\boldsymbol{Imag}\sum_{k=1}^{4}\boldsymbol{Res}(\psi_{u2}(z_k))}{ab\{(a^2\cos^2\varphi'+b^2\sin^2\varphi')t_2'^2+2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t_2'+a^2\sin^2\varphi'+b^2\cos^2\varphi'\}^2}\Bigg]\\
v &= \frac{8p_0}{3(\lambda+\mu)}\cdot\frac{\delta^2\nu}{(\nu-\delta)}\\
&\quad\times\Bigg[\frac{abA\ \boldsymbol{Imag}\sum_{k=1}^{4}\boldsymbol{Res}(\psi_{v1}(z_k))}{(a^2\cos^2\varphi'+b^2\sin^2\varphi')t_2'^2+2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t_2'+a^2\sin^2\varphi'+b^2\cos^2\varphi'}\\
&\quad+\frac{\boldsymbol{Imag}\sum_{k=1}^{4}\boldsymbol{Res}(\psi_{v2}(z_k))}{ab\{(a^2\cos^2\varphi'+b^2\sin^2\varphi')t_2'^2+2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t_2'+a^2\sin^2\varphi'+b^2\cos^2\varphi'\}^2}\Bigg]
\end{aligned}\right\} \tag{17}$$

in which

$$\boldsymbol{Res}(\psi_{u1}(s_1)) = \frac{s_1 B_1(s_1)}{2(s_1^2-\bar{s}_1^2)B_4(s_1)}, \quad \boldsymbol{Res}(\psi_{u1}(s_2)) = \frac{\bar{s}_1 B_1(\bar{s}_1)}{2(s_1^2-\bar{s}_1^2)B_4(\bar{s}_1)}$$

$$\boldsymbol{Res}(\psi_{u1}(s_3)) = \frac{s_3 B_1(s_3)}{2(s_3^2-\bar{s}_3^2)B_3(s_3)}, \quad \boldsymbol{Res}(\psi_{u1}(s_4)) = \frac{\bar{s}_3 B_1(\bar{s}_3)}{2(s_3^2-\bar{s}_3^2)B_3(\bar{s}_3)}$$

$$\boldsymbol{Res}(\psi_{u2}(s_1)) = \frac{s_1 B_1(s_1)\{B_2(s_1)\}^2}{2(s_1^2-\bar{s}_1^2)\{B_4(s_1)\}^2}, \quad \boldsymbol{Res}(\psi_{u2}(s_2)) = \frac{\bar{s}_1 B_1(\bar{s}_1)\{B_2(\bar{s}_1)\}^2}{2(s_1^2-\bar{s}_1^2)\{B_4(\bar{s}_1)\}^2}$$

$$\boldsymbol{Res}(\psi_{u2}(s_3)) = \frac{B_2(s_3)}{4s_3(s_3^2-\bar{s}_3^2)^3\{B_3(s_3)\}^2}$$
$$\times\Big[2(s_3^2-\bar{s}_3^2)B_3(s_3)\{B_1(s_3)B_2(s_3)+(\cos\varphi'-\sin\varphi'\cdot t_2')s_3^2B_2(s_3)+2s_3^2B_1(s_3)$$
$$\times\big((b^2x\sin\varphi'-a^2y\cos\varphi')t_2'-b^2x\cos\varphi'-a^2y\sin\varphi'\big)\}$$
$$-B_1(s_3)B_2(s_3)\Big\{4\,s_3^2\Big(s_3^2+\frac{(\nu+\delta)\delta}{\nu-\delta}\Big)(s_3^2-\bar{s}_3^2)+(5s_3^2-\bar{s}_3^2)B(s_3)\Big\}\Big]$$

$$\boldsymbol{Res}(\psi_{u2}(s_4)) = \frac{-B_2(\bar{s}_3)}{4\bar{s}_3(s_3^2-\bar{s}_3^2)^3\{B_3(\bar{s}_3)\}^2}$$
$$\times\Big[2(s_3^2-\bar{s}_3^2)B_3(\bar{s}_3)\{B_1(\bar{s}_3)B_2(\bar{s}_3)+(\cos\varphi'-\sin\varphi'\cdot t_2')\bar{s}_3^2B_2(\bar{s}_3)$$
$$+2\bar{s}_3^2B_1(\bar{s}_3)\big((b^2x\sin\varphi'-a^2y\cos\varphi')t_2'-b^2x\cos\varphi'-a^2y\sin\varphi'\big)\}$$
$$+B_1(\bar{s}_3)B_2(\bar{s}_3)\Big\{-4\bar{s}_3^2\Big(\bar{s}_3^2+\frac{(\nu+\delta)\delta}{\nu-\delta}\Big)(s_3^2-\bar{s}_3^2)+(5\bar{s}_3^2-s_3^2)B_3(\bar{s}_3)\Big\}\Big]$$

$$\boldsymbol{Res}(\psi_{v1}(s_1)) = \frac{s_1 B_0(s_1)}{2(s_1^2-\bar{s}_1^2)B_4(s_1)}, \quad \boldsymbol{Res}(\psi_{v1}(s_2)) = \frac{\bar{s}_1 B_0(\bar{s}_1)}{2(s_1^2-\bar{s}_1^2)B_4(\bar{s}_1)}$$

$$\boldsymbol{Res}(\psi_{v1}(s_3)) = \frac{s_3 B_0(s_3)}{2(s_3^2-\bar{s}_3^2)B_3(s_3)}, \quad \boldsymbol{Res}(\psi_{v1}(s_4)) = \frac{\bar{s}_3 B_0(\bar{s}_3)}{2(s_3^2-\bar{s}_3^2)B_3(\bar{s}_3)}$$

$$\boldsymbol{Res}(\psi_{v2}(s_1)) = \frac{s_1 B_0(s_1)\{B_2(s_1)\}^2}{2(s_1^2-\bar{s}_1^2)\{B_4(s_1)\}^2}, \quad \boldsymbol{Res}(\psi_{v2}(s_2)) = \frac{\bar{s}_1 B_0(\bar{s}_1)\{B_2(\bar{s}_1)\}^2}{2(s_1^2-\bar{s}_1^2)\{B_4(\bar{s}_1)\}^2}$$

$$\boldsymbol{Res}(\psi_{v2}(s_3)) = \frac{B_2(s_3)}{4s_3(s_3^2-\bar{s}_3^2)^3\{B_3(s_3)\}^2}$$
$$\times\Big[2(s_3^2-\bar{s}_3^2)B_3(s_3)\{B_0(s_3)B_2(s_3)+(\cos\varphi'\cdot t_2'+\sin\varphi')s_3^2B_2(s_3)$$
$$+2s_3^2B_0(s_3)((b^2x\sin\varphi'-a^2y\cos\varphi')t_2'-b^2x\cos\varphi'-a^2y\sin\varphi')\}$$
$$-B_0(s_3)B_2(s_3)\Big\{4s_3^2\Big(s_3^2+\frac{(\nu+\delta)\delta}{\nu-\delta}\Big)(s_3^2-\bar{s}_3^2)+(5s_3^2-\bar{s}_3^2)B_3(s_3)\Big\}\Big]$$

$$\boldsymbol{Res}(\psi_{v2}(s_4)) = \frac{-B_2(\bar{s}_3)}{4\bar{s}_3(s_3^2-\bar{s}_3^2)\{B_3(\bar{s}_3)\}^2}$$
$$\times\Big[2(s_3^2-\bar{s}_3^2)B_3(\bar{s}_3)\{B_0(\bar{s}_3)B_2(\bar{s}_3)+(\cos\varphi'\cdot t_2'+\sin\varphi')\bar{s}_3^2B_2(\bar{s}_3)$$
$$+2\bar{s}_3^2B_0(\bar{s}_3)\big((b^2x\sin\varphi'-a^2y\cos\varphi')\,t_2'-b^2x\,\mathrm{cso}\varphi'-a^2y\sin\varphi'\big)\}$$
$$+B_0(\bar{s}_3)B_2(\bar{s}_3)\Big\{-4\bar{s}_3^2\Big(\bar{s}_3^2+\frac{(\nu+\delta)\delta}{\nu-\delta}\Big)(s_3^2-\bar{s}_3^2)+(5\bar{s}_3^2-s_3^2)B_3(\bar{s}_3)\Big\}\Big]$$

$$B_0(z) = (t_2'\cos\varphi' + \sin\varphi')z^2 - \delta(t_1'\cos\varphi' + \sin\varphi')$$

$$B_1(z) = (\cos\varphi' - t_2'\sin\varphi')z^2 + \delta(t_2'\sin\varphi' - \cos\varphi')$$

$$B_2(z) = \{(b^2x\sin\varphi' - a^2y\cos\varphi')t_2' - (b^2x\cos\varphi' + a^2y\sin\varphi')\}z^2 - \delta\{(b^2x\sin\varphi' - a^2y\cos\varphi')t_1' - b^2x\cos\varphi' - a^2y\sin\varphi'\}$$

$$B_3(z) = z^4 + \frac{2(\nu+\delta)\delta}{\nu-\delta}z^2 + \delta^2$$

$$B_4(z) = z^4 + 2Pz^2 + Q$$

$$P = -\frac{\delta\{(a^2\cos^2\varphi' + b^2\sin^2\varphi')t_1't_2' + a^2\sin^2\varphi' + b^2\cos^2\varphi'\}}{(a^2\cos^2\varphi' + b^2\sin^2\varphi')t_2'^2 + 2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t_2' + a^2\sin^2\varphi' + b^2\cos^2\varphi'}$$

$$Q = \frac{\delta^2\{(a^2\cos^2\varphi' + b^2\sin^2\varphi')t_1'^2 + 2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t_1' + a^2\sin^2\varphi' + b^2\cos^2\varphi'\}}{(a^2\cos^2\varphi' + b^2\sin^2\varphi')t_2'^2 + 2(a^2-b^2)\sin\varphi'\cos\varphi'\cdot t_2' + a^2\sin^2\varphi' + b^2\cos^2\varphi'}$$

Singular points s_1, s_2, s_3 and s_4 are given as follows;

$$s_1 = -\sqrt{\frac{-\nu\delta}{\nu-\delta}} + \sqrt{\frac{\delta^2}{\nu-\delta}}\,i\,, \qquad s_2 = \sqrt{\frac{-\nu\delta}{\nu-\delta}} + \sqrt{\frac{\delta^2}{\nu-\delta}}\,i$$

$$s_3 = -\sqrt{\frac{\sqrt{Q}-P}{2}} + \sqrt{\frac{\sqrt{Q}+P}{2}}\,i\,, \qquad s_4 = \sqrt{\frac{\sqrt{Q}-P}{2}} + \sqrt{\frac{\sqrt{Q}+P}{2}}\,i$$

iii) For a point $(x, y, 0)$ on a line $x=a$

Displacements u and v are readily obtained in the before-mentioned manner

$$\left.\begin{aligned} u &= -\frac{p_0}{4(\lambda+\mu)}\Big[-\frac{2ay^2}{3b(a^2-b^2)}\{2a-\sqrt{2(\sqrt{q_1}-p_1)}\} + \frac{2a^2}{3b(a^2-b^2)^2} \\ &\quad\times\Big\{\{(b^4-a^2y^2)\sqrt{2(\sqrt{q_1}-p_1)} - 2ab^2y\sqrt{2(\sqrt{q_1}+p_1)}\}/a + 2a^2y^2 + a^2b^2 + 4b^2y^2 - 3b^4\Big\}\Big] \\ v &= -\frac{p_0}{4(\lambda+\mu)}\Big[\frac{2ay^2}{3b(a^2-b^2)}\{\sqrt{2(\sqrt{q_1}+p_1)} - 2y\} - \frac{2a}{3b(a^2-b)^2}\{2ab^2y\sqrt{2(\sqrt{q_1}-p_1)} \\ &\quad+(b^4-a^2y^2)\sqrt{2(\sqrt{q_1}+p_1)} + y(3a^2b^2 - 2a^2y^2 + 3b^4)\}\Big] \end{aligned}\right\}(18)$$

where

$$p_1 = y^2 - b^2\,, \quad q_1 = (y^2-b^2)^2 + 4a^2y^2$$

3.3.3. Vertical displacement at a point $(x, y, 0)$ within the elliptical area

From the third of eqs. (17), we find

$$w = \frac{(\lambda+2\mu)p_0}{4\pi\mu(\lambda+\mu)}\Big[\frac{a^2b^2 - b^2x^2 - a^2y^2}{ab}\int_0^{2\pi}\frac{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta - (x\sin\theta - y\cos\theta)^2}}{a^2\sin^2\theta + b^2\cos^2\theta}d\theta$$

$$+\frac{1}{ab}\int_0^{2\pi}\frac{(b^2x\cos\theta + a^2y\sin\theta)^2\sqrt{a^2\sin^2\theta + b^2\cos^2\theta - (x\sin\theta - y\cos\theta)^2}}{(a^2\sin^2\theta + b^2\cos^2\theta)^2}d\theta$$

$$-\frac{ab}{3}\int_0^{2\pi}\frac{\{a^2\sin^2\theta + b^2\cos^2\theta - (x\sin\theta - y\cos\theta)^2\}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta - (x\sin\theta - y\cos\theta)^2}}{(a^2\sin^2\theta + b^2\cos^2\theta)^2}d\theta\Big] \quad (19)$$

Now we consider the displacement w at a point on the x- and y-axis.

(I) Displacement w on the x-axis

Substituting $y=0$ into eq. (19), we obtain

$$w = \frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left[\frac{(a^2-x^2)b}{a}\int_0^{\pi/2}\frac{\sqrt{(a^2-x^2)\sin^2\theta+b^2\cos^2\theta}}{a^2\sin^2\theta+b^2\cos^2\theta}d\theta\right.$$
$$-\frac{ab}{3}(a^2-x^2)\int_0^{\pi/2}\frac{\sqrt{(a^2-x^2)\sin^2\theta+b^2\cos^2\theta}}{(a^2\sin^2\theta+b^2\cos^2\theta)^2}d\theta$$
$$\left.+\frac{(3b^3x^2-a^2b^3+a^4b-a^2bx^2)}{3a}\int_0^{\pi/2}\frac{\cos^2\theta\sqrt{(a^2-x^2)\sin^2\theta+b^2\cos^2\theta}}{(a^2\sin^2\theta+b^2\cos^2\theta)^2}d\theta\right] \tag{20}$$

For the calculation of eq. (20), we consider the following cases depending on the value of x.

i) The case $|x|<ea$

Putting $k^2=1-\dfrac{b^2}{a^2-x^2}$, eq. (20) is written as

$$w = \frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left[\frac{b(a^2-x^2)\sqrt{a^2-x^2}}{a^3}\int_0^{\pi/2}\frac{\sqrt{1-k^2\cos^2\theta}}{1-e^2\cos^2\theta}d\theta\right.$$
$$-\frac{b(a^2-x^2)\sqrt{a^2-x^2}}{3a^3}\int_0^{\pi/2}\frac{\sqrt{1-k^2\cos^2\theta}}{(1-e^2\cos^2\theta)^2}d\theta$$
$$\left.+\frac{b(3b^2x^2-a^2b^2+a^4-a^2x^2)\sqrt{a^2-x^2}}{3a^5}\int_0^{\pi/2}\frac{\cos^2\theta\sqrt{1-k^2\cos^2\theta}}{(1-e^2\cos^2\theta)^2}d\theta\right] \tag{21}$$

ii) The case $|x|=ea$

Eq. (20) is easily calculated as

$$w = \frac{(\lambda+2\mu)p_0b}{2\mu(\lambda+\mu)}\left(\frac{2}{3}-\frac{x^2}{3a^2}\right) \tag{22}$$

iii) The case $|x|\geqq ea$

Putting $k'^2 = 1-\dfrac{a^2-x^2}{b^2}$, eq. (20) is given as

$$w = \frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left[\frac{b^2(a^2-x^2)}{a^3}\int_0^{\pi/2}\frac{\sqrt{1-k'^2\sin^2\theta}}{1-e^2\cos^2\theta}d\theta-\frac{b^2(a^2-x^2)}{3a^3}\int_0^{\pi/2}\frac{\sqrt{1-k'^2\sin^2\theta}}{(1-e^2\cos^2\theta)^2}d\theta\right.$$
$$\left.+\frac{b^2(3b^2x^2-a^2b^2+a^4-a^2x^2)}{3a^5}\int_0^{\pi/2}\frac{\cos^2\theta\sqrt{1-k'^2\sin^2\theta}}{(1-e^2\cos^2\theta)^2}d\theta\right] \tag{23}$$

(II) Displacement w on the y-axis

Substituting $x=0$ into eq. (19), we obtain

$$w=\frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left[\frac{a(b^2-y^2)}{b}\int_0^{\pi/2}\frac{\sqrt{a^2\sin^2\theta+(b^2-y^2)\cos^2\theta}}{a^2\sin^2\theta+b^2\cos^2\theta}d\theta+\frac{a^3(3y^2-b^2)}{3b}\int_0^{\pi/2}\frac{\sqrt{a^2\sin^2\theta+(b^2-y^2)\cos^2\theta}}{(a^2\sin^2\theta+b^2\cos^2\theta)^2}d\theta-\frac{a(3a^2y^2-a^2b^2+b^4-b^2y^2)}{3b}\times\int_0^{\pi/2}\frac{\sqrt{a^2\sin^2\theta+(b^2-y^2)\cos^2\theta}\cos^2\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)^2}d\theta\right] \tag{24}$$

Putting $K^2=1-\dfrac{b^2-y^2}{a^2}$, eq. (24) is expressed as follows;

$$w=\frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left[\frac{b^2-y^2}{b}\int_0^{\pi/2}\frac{\sqrt{1-K^2\cos^2\theta}}{1-e^2\cos^2\theta}d\theta+\frac{(3y^2-b^2)}{3b}\int_0^{\pi/2}\frac{\sqrt{1-K^2\cos^2\theta}}{(1-e^2\cos^2\theta)^2}d\theta-\frac{(3a^2y^2-a^2b^2+b^4-b^2y^2)}{3b}\int_0^{\pi/2}\frac{\cos^2\theta\sqrt{1-K^2\cos^2\theta}}{(1-e\cos^2\theta)^2}d\theta\right] \tag{25}$$

3.3.4 Vertical displacement at a point $(x, y, 0)$ without the elliptical area

From the third of eqs. (8), we find

$$w=\frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left[\frac{1}{3ab}\int_{\theta_1}^{\theta_2}\frac{(b^2x\cos\theta+a^2y\sin\theta)^2\sqrt{a^2\sin^2\theta+b^2\cos^2\theta-(x\sin\theta-y\cos\theta)^2}}{(a^2\sin^2\theta+b^2\cos^2\theta)^2}d\theta-\frac{ab}{3}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\int_{\theta_1}^{\theta_2}\frac{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta-(x\sin\theta-y\cos\theta)^2}}{a^2\sin^2\theta+b^2\cos^2\theta}d\theta\right] \tag{26}$$

(I) Displacement w on the x-axis

Substituting $y=0$ into eq. (26) and putting

$$E^2=1+\frac{x^2-a^2}{b^2},$$

we obtain

$$w=\frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left\{\frac{2b^4x^2}{3a^5}\int_{\theta_1}^{\theta_2}\frac{\cos^2\theta\sqrt{1-E^2\sin^2\theta}}{(1-e^2\cos^2\theta)^2}d\theta-\frac{2b^2}{3a}\left(\frac{x^2}{a^2}-1\right)\int_{\theta_1}^{\theta_2}\frac{\sqrt{1-E^2\sin^2\theta}}{1-e^2\cos^2\theta}d\theta\right\} \tag{27}$$

Replacing $E\sin\theta$ with $\sin\rho$, eq. (27) becomes

$$w=\frac{(\lambda+2\mu)p_0}{\mu\pi(\lambda+\mu)}\left[\frac{2b^4E^3}{3a^3e^2}\left\{\int_0^{\pi/2}\frac{\sqrt{1-\frac{1}{E^2}\sin^2\rho}}{\left(1-\frac{a^2e^2}{x^2}\cos^2\rho\right)^2}d\rho-\int_0^{\pi/2}\frac{\sqrt{1-\frac{1}{E^2}\sin^2\rho}}{1-\frac{a^2e^2}{x^2}\cos^2\rho}d\rho\right\}-\frac{2b^2}{3a}\left(\frac{x^2}{a^2}-1\right)\frac{E}{e^2}\left\{\int_0^{\pi/2}\frac{1}{\left(1-\frac{a^2e^2}{x^2}\cos^2\rho\right)\sqrt{1-\frac{1}{E^2}\sin^2\rho}}d\rho-\int_0^{\pi/2}\frac{1}{\sqrt{1-\frac{1}{E^2}\sin^2\rho}}d\rho\right\}\right] \tag{27'}$$

(II) Displacement w on the y-axis

Substituting $x=0$ into eq. (26) and replacing θ with $\alpha-\frac{\pi}{2}$, eq. (26) is written as

$$w=\frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left[\begin{array}{l}\dfrac{a^3y^2}{3b}\displaystyle\int_{\alpha_1}^{\alpha_2}\frac{\cos^2\alpha\sqrt{a^2\cos^2\alpha+(b^2-y^2)\sin^2\alpha}}{(a^2\cos^2\alpha+b^2\sin^2\alpha)^2}d\alpha\\ -\dfrac{ab}{3}\left(\dfrac{y^2}{b^2}-1\right)\displaystyle\int_{\alpha_1}^{\alpha_2}\frac{\sqrt{a^2\cos^2\alpha+(b^2-y^2)\sin^2\alpha}}{a^2\cos\alpha^2+b^2\sin^2\alpha}d\alpha\end{array}\right] \tag{28}$$

in which

$$\alpha_2=-\alpha_1=\tan^{-1}\frac{a}{\sqrt{y^2-b^2}}$$

Putting $E'^2=1+\dfrac{y^2-b^2}{a^2}$ and $E'\sin\alpha=\sin\gamma$,

we have

$$w=\frac{(\lambda+2\mu)p_0}{\pi\mu(\lambda+\mu)}\left[\frac{2y^2E'}{3b\,e^2}\left\{\left(\frac{e^2}{E'^2}-1\right)\int_0^{\pi/2}\frac{\sqrt{1-\frac{1}{E'^2}\sin^2\gamma}}{\left(1-\frac{e^2}{E'^2}\sin^2\gamma\right)}d\gamma+\int_0^{\pi/2}\frac{\sqrt{1-\frac{1}{E'^2}\sin^2\gamma}}{1-\frac{e^2}{E'^2}\sin^2\gamma}d\gamma\right\}\right.$$

$$-\frac{2b}{3}\left(\frac{y^2}{b^2}-1\right)\frac{E'}{e^2}\left\{\left(\frac{e^2}{E'^2}-1\right)\int_0^{\pi/2}\frac{1}{\left(1-\frac{e^2}{E'^2}\sin^2\gamma\right)\sqrt{1-\frac{1}{E'^2}\sin^2\gamma}}d\gamma\right.$$

$$\left.\left.+\int_0^{\pi/2}\frac{1}{\sqrt{1-\frac{1}{E'^2}\sin^2\gamma}}d\gamma\right\}\right] \tag{28$'$}$$

3.3.5 Special case

When the loaded area is circular, u, v, w are easily obtained with polar coordinates, as the special case of the elliptical aιea.

In this case displacements are symmetrical with respect to the z-axis, so we have only to consider radial and vertical displacements U and W respectively.

(I) Displacement within the circular area

i) Radial displacement U

After calculation we obtain

$$U=-\frac{p_0}{2(\lambda+\mu)}\left(\frac{R_0}{2}-\frac{R_0^3}{4a_0^2}\right) \tag{29}$$

where a_0 is the radius of circle and R_0 is the distance of any point from origin.

ii) Vertical displacement W

From the third of eqs. (7), we have

$$W=\frac{(\lambda+2\mu)p_0}{2\pi\mu(\lambda+\mu)}\left\{\frac{4}{3}\left(1-\frac{R_0^2}{a_0^2}\right)\int_0^{\pi/2}\sqrt{a_0^2-R_0^2\sin^2\theta}\,d\theta\right.$$

$$+\frac{4}{3}\frac{R_0^2}{a_0^2}\int_0^{\pi/2}\cos^2\theta\sqrt{a_0^2-R_0^2\sin^2\theta}\,d\theta\Big\} \tag{30}$$

(II) Displacements without the circular area

i) Radial displacement U

After calculation we obtain

$$U=-\frac{p_0}{2(\lambda+\mu)}\frac{a_0^2}{R_0} \tag{31}$$

ii) Vertical displacement W

$$W=\frac{(\lambda+2\mu)p_0}{4\pi\mu(\lambda+\mu)}\Big\{\frac{8}{3}\Big(1-\frac{R_0^2}{a_0^2}\Big)\frac{a_0^2}{R_0}\int_0^{\pi/2}\frac{\cos^2\rho}{\sqrt{1-\frac{a_0^2}{R_0^2}\sin^2\rho}}\,d\rho$$

$$+\frac{8}{3}R_0\int_0^{\pi/2}\cos^2\rho\sqrt{1-\frac{a_0^2}{R_0^2}\sin^2\rho}\,d\rho\Big\} \tag{32}$$

4. Results

The displacements obtained at a point on the x- and y-axis for various values of e are shown in Figs. 3–6. However, the similar tendencies are recognized at any other point on the surface of the semi-infinite solid.

In plotting the results the following non-dimensional quantities are used:

Horizontal displacement in the x-direction: $\frac{2(\lambda+\mu)}{ap_0}u$; y-direction: $\frac{2(\lambda+\mu)}{bp_0}v$

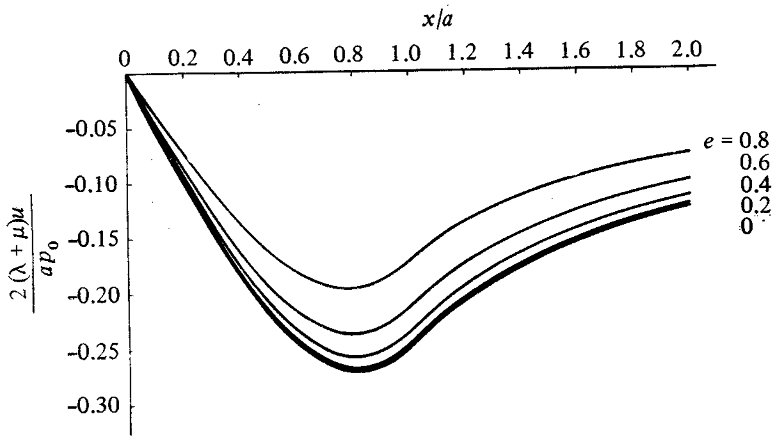

Fig. 3. Horizontal displacement on the x-axis

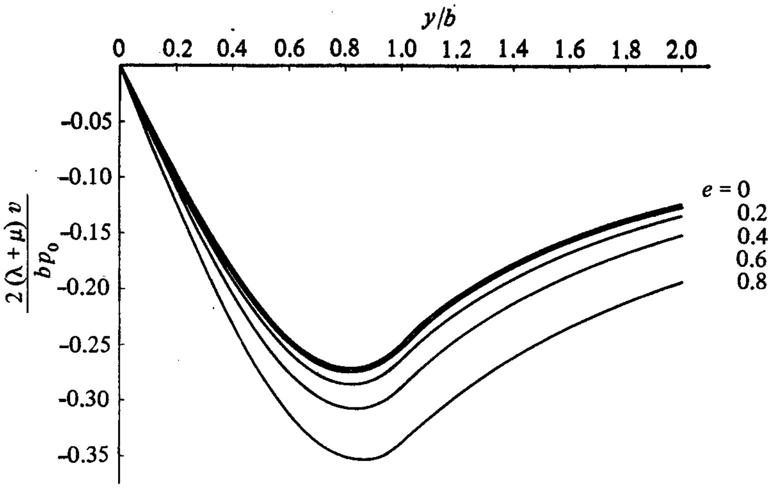

Fig. 4. Horizontal displacement on the y-axis.

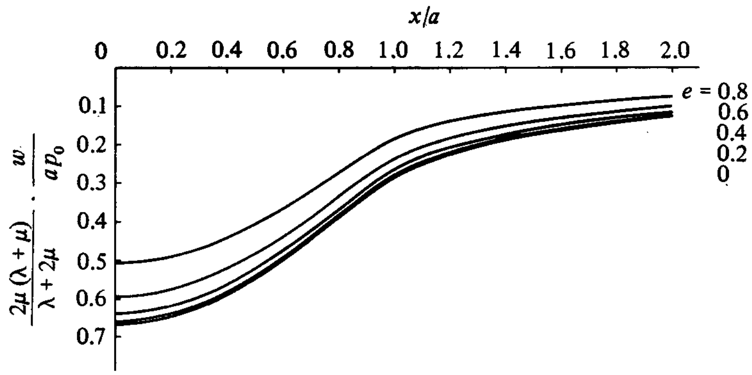

Fig. 5. Vertical displacement on the x-axis.

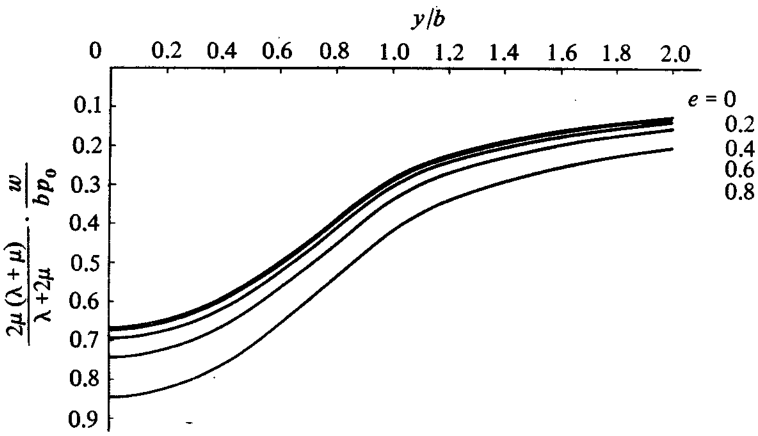

Fig. 6. Vertical displacement on the y-axis.

Vertical displacement on the
- x-axis $\dfrac{2\mu(\lambda+\mu)}{\lambda+2\mu}\dfrac{1}{ap_0}w$
- y-axis $\dfrac{2\mu(\lambda+\mu)}{\lambda+2\mu}\dfrac{1}{bp_0}w$

abscissa: $\dfrac{x}{a}$, ordinate: $\dfrac{y}{b}$

4.1 Horizontal displacements

It will be seen that displacement u directs inwards at every point on the surface. It increases almost lineally to its maximum value at the inner point near the boundary, then diminishes gradually with the increase of x/a. Although the same tendency is found for displacement v, the positions of the maximum values for u and v move to the opposite direction with the increase of e, that is, the former moves rightwards, to the contrary the latter leftwards.

4.2 Vertical displacement

As shown in Figs. 5 and 6 displacement w has a maximum value in the center of ellipse and gradually decreases in the concave form with the increase of x/a, turning to the convex form, as the value of x/a approaches 1.

5. Experiment

The following experiments were performed to ascertain the theoretical results.

5.1 Experimental apparatus

This apparatus consists of two main parts, one having 7 pressure sensor pins, another the same number of displacements indicator pins (Fig. 7, Fig. 8).

To apply the pressure we have used a moving platen having the plane elliptical surface, the eccentricity of which is about 0.745. A small pin, fitted vertically into the moving platen and ending flush with its surface, makes contact with the load cell, as a

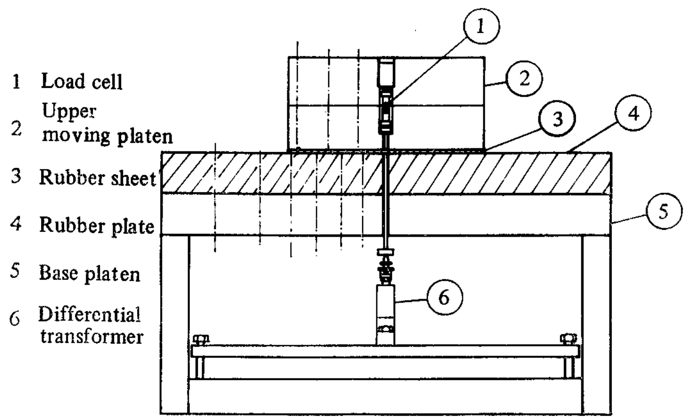

Fig. 7. Skeleton of the experimental apparatus.

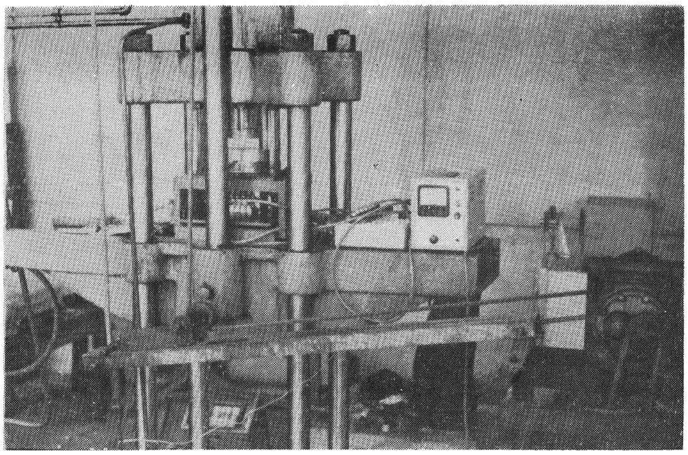

Fig. 8. Photograph showing the apparatus mounting on the Amsler testing machine.

kind of pressure sensor which provides a measure of the load compressing the surface of the material.

The vertical displacement of the surface of material is measured with the differential transformer through a pin which is inserted vertically into the material and ends at the same level as its surface. This apparatus is set in the supporting frame.

5.2 Experimental procedure

We need too high pressure to obtain experimentally the vertical displacement of the surface of metal in the measurable order, so the rubber plate having the finite thickness was used instead of a semi-infinite solid to multiply the values to be measured. But Young's modulus of rubber is extremely small in comparison with metal, so we need to use the sensitive load cell to measure the feeble pressure corresponding the vertical displacement. For this purpose we have adopted the cylindrical rubber rod as the sensor, on the surface of which the wire strain gauges are attached.

Firstly we adhere the rubber plate on the lower base platen, under which the differential transformers measuring the displacements are set. The upper moving platen having load cell calibrated previously is placed upon the center of the base platen. On the process developing the predetermined form of pressure distribution, we coated the surface of the upper elliptical moving plate with the thin rubber sheet having the various type of a section of convex lens. In this experiment, firstly the rubber sheet with uniform thickness have been used. To eliminate the tangential force the soapy lubricant is applied between the contact surface of the rubber sheet and the rubber plate.

Under the small load we have made the zero-adjustment of the load cells and differential transformers. After finishing adjustment, pressure and displacement curves are obtained with increasing the load stepwise.

5.3 Experimental results

The measured displacement is shown in Fig. 9. For the comparison of a tendency

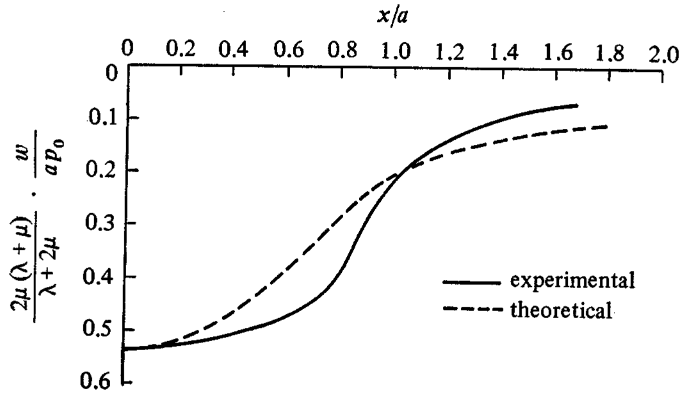

Fig. 9. Vertical displacement.
Experimental result refers to the mean pressure p_0
Maximum value (at $x=0$): 0.29μ
Total load: 1.9t

of the experimental curve with the theoretical one, the latter is drawn with the dotted line. Each curve was plotted so as to have the same value in the center of the elliptical region. As seen from their figures for the inner region, the experimental values are larger than the theoretical, while for the outer one the opposite tendency is observed, which seems to be due to the character of rubber.

6. Conclusion

The vertical displacement of the surface compressed with a distributed pressure over an elliptical area was obtained experimentally. However its data is somewhat scattered, besides the form of the predetermined pressure distribution was not successfully reproduced owing to the difficulty of the experimental technique, because the pressure sensor used in this experiment is sensitive to disturbances and the balancing point is changeable with time. Therefore the precise comparison of the results obtained experimentally and theoretically could be scarcely desired. We found that it is very difficult to perform the experiment using rubber, since its behaviour of deformation is much complicated.

In this paper the friction acting on the surface of contact is ignored, which must be considered practically. The solutions of the more practical contact problem considering the frictional force will be obtained, if we superpose the solutions of the semi-infinite solid under the tangential surface tractions upon those given in this paper. When the pressure measured does not fall to zero at the boundary of the loaded area, we have only to superpose the solutions[3] under the uniform pressure upon the present ones.

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