On a Solution of Stochastic Nonlinear Programming Problems

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There are many engineering problems which are reduced to the mathematical programming problem. The constraints and/or the objective function established are sometimes subjected to errors due to experiments or estimations, and thus they are probabilistic in nature. In such a case, a stochastic approach must be adopted to make the program realistic by treating the constraints and/or the objective function as random variables. Thus we set up the problems 1) to minimize the expected value of the objective function under the chance-constraints on the constriants and 2) to minimize the expected value of the objective function under the chance-constraint on the objective function as well as those on the constraints.

The constraints and the objective function are random variables, the distributions of which are not predetermined. Thus, the chance-constraints are not to be calculated directly. In this paper, a unique approach is employed to transform those into the equivalent deterministic nonlinear constraints. Validity of this transformation is proved by using Tchebychev inequality. A possible algorithm to solve the problems is proposed and numerical examples are also provided to illustrate the given method.

1. Introduction

There are many engineering problems which are reduced to those of the mathematical programming. The constraints and/or the objective function may be sometimes deterministic in nature, but there are some other cases when those are modelled by experiments[1] or estimations. In such cases, they are subjected to errors and thus the solution obtained by using them may be optimal for the particular case, e.g., for the mean value, but it is not so for general cases. It should be also noted that since the optimal solution usually lies on the boundary of the constraints, the active constraints which are best fit, for example, in the sense of mean are not satisfied with probability 0.5, approximately. Further, when the costs or the profits are selected as the objective function, they may sometimes deviate too much to complete the program. In order to make the programs in these cases realistic, a probabilistic approach must be applied, and thus the constraints and the objective function are to be treated as random variables.

Fairly systematic researches have been made on the linear programming problems with uncertainty[2)~6)]. Those are categorized in the following three: 1) replacing the random elements by their expected values, 2) replacing the random elements by pessimistic estimates of their values and 3) recasting the problem into a two-stage problem where,

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in the second stage, one can compensate for inaccuracies in the first stage activities. Little has been done on a nonlinear programming problems with uncertainty[7)~9)].

This paper is concerned with the formulation and solution of the stochastic nonlinear programming problems. It is shown that the problems are reduced to the deterministic nonlinear programming problems. Numerical examples are provided to illustrate the procedure and its usefulness.

Nomenclatures

$\overline{(\cdot)}$: mean value of $(\cdot)$ $\sigma_{(\cdot)}$: standard deviation of $(\cdot)$

$\Delta_{(\cdot)} \equiv (\cdot) - \overline{(\cdot)}$: deviation of $(\cdot)$ from its mean value

Prob. $[(\cdot)]$: probability of the event $(\cdot)$

x: control vector, the elements of which are x_i's

a, c: coefficient vectors, the elements of which are a_i's, c_i's

$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$: standard Gaussian probability density function

2. Statement of the Problem

Consider the standard nonlinear programming problem:
Under the constraints

$$g_i(x, a) \geq 0 \qquad (i = 1, 2, \cdots, m) \tag{1}$$

find the control vector x^* to minimize the objective function

$$z = f(x, c) \tag{2}$$

In the above, we use the notations

$x = \text{col.}\ (x_i) =$ n-dimensional control vector
$a = \text{col.}\ (a_i) =$ q-dimensional coefficient vector
$c = \text{col.}\ (c_i) =$ r-dimensional coefficient vector

and assume that $g_i(\cdot, \cdot\cdot)$ and $f(\cdot, \cdot\cdot)$ are real valued functions with sufficient smoothness.

Let us now consider the case when the coefficient vectors a, c are random variables with known probability distributions. Then, we set up the following problems:
Problem 1 "Under the chance-constraints:

$$\text{Prob.}\ [g_i(x, a) \geq 0] \geq P_i \qquad (i = 1, 2, \cdots, m) \tag{3}$$

find x^* to minimize the expected value of the objective function $\bar{z}$, where P_i's are given constants."

Problem 2 "Under the constraints (3) and an additional chance-constraint on the deviation of the objective function:

$$\text{Prob.}\ [z \geq \beta \bar{z}] \leq P_z \tag{4}$$

find x^* to minimize $\bar{z}$, where β and P_z are given constants."

For the problems to have the solution, we assume the following: First we define the sets

$$X_i = \{x \mid \text{Prob.}\ [g_i(x, a) \geq 0] \geq P_i\} \tag{5}$$

$$X_z = \{x \mid \text{Prob.}\ [z \geq \beta \bar{z}] \leq P_z\} \tag{6}$$

For the problem 1, we assume that the set

$$X_c = \bigcap_{i=1}^{m} X_i \tag{7}$$

is not empty. For the problem 2, we assume that the set

$$X_t = X_c \cap X_z \tag{8}$$

is not empty.

3. Formulation of the Problem

If we know the analytical expression of the distribution of the constraints and the objective function, we can formulate the chance-constraints (3) and (4). This, however, can not be done in general. In this paper, we evaluate them by the equivalent nonlinear constraints as mentioned below.

First, we calculate the mean and the variance of $g_i(x, a)$. Expanding $g_i(x, a)$ into Taylor's series about the mean value $a=\bar{a}$ yields

$$g_i(x, a) = g_i(x, \bar{a}) + \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{j=1}^{q} \frac{\partial}{\partial a_j} \Delta a_j \right)^m g_i(x, a) \tag{9}$$

where $\Delta a_j \equiv a_j - \bar{a}_j$ and the partial derivatives are evaluated for the mean value $a=\bar{a}$.

Thus, the mean of $g_i(x, a)$ is given by

$$\begin{aligned}
\overline{g_i(x, a)} &= g_i(x, \bar{a}) + \sum_{m=2}^{\infty} \frac{1}{m!} \overline{\left(\sum_{j=1}^{q} \frac{\partial}{\partial a_j} \Delta a_j \right)^m g_i(x, a)} \\
&= g_i(x, \bar{a}) + \frac{1}{2} \sum_{j,k=1}^{q} \left(\frac{\partial^2 g_i}{\partial a_j \partial a_k} \right) \overline{\Delta a_j \Delta a_k} + \frac{1}{6} \sum_{j,k,l=1}^{q} \left(\frac{\partial^3 g_i}{\partial a_i \partial a_k \partial a_1} \right) \overline{\Delta a_j \Delta a_k \Delta a_1} \\
&\quad + \frac{1}{24} \sum_{j,k,l,m=1}^{q} \left(\frac{\partial^4 g}{\partial a_j \partial a_k \partial a_l \partial a_m} \right) \overline{\Delta a_j \Delta a_k \Delta a_l \Delta a_m} + \cdots
\end{aligned} \tag{10}$$

The variance is

$$\begin{aligned}
\sigma_{g_i}^2 &= \overline{\{g_i(x, a) - \overline{g_i(x, a)}\}^2} \\
&= \overline{\left[\sum_{m=1}^{\infty} \frac{1}{m!} \left\{ \left(\sum_{j=1}^{q} \frac{\partial}{\partial a_j} \Delta a_j \right)^m g_i - \overline{\left(\sum_{j=1}^{q} \frac{\partial}{\partial a_j} \Delta a_j \right)^m g_i} \right\} \right]^2} \\
&= \sum_{j,k=1}^{q} \left(\frac{\partial g_i}{\partial a_j} \right) \left(\frac{\partial g_i}{\partial a_k} \right) \overline{\Delta a_j \Delta a_k} + \sum_{j,k,l=1}^{q} \left(\frac{\partial g_i}{\partial a_j} \right) \left(\frac{\partial^2 g_i}{\partial a_k \partial a_l} \right) \overline{\Delta a_j \Delta a_k \Delta a_l} \\
&\quad + \frac{1}{12} \sum_{j,k,l,m=1}^{q} \left\{ 3 \left(\frac{\partial^2 g_i}{\partial a_j \partial a_k} \right) \left(\frac{\partial^2 g_i}{\partial a_l \partial a_m} \right) + 4 \left(\frac{\partial g_i}{\partial a_j} \right) \left(\frac{\partial^3 g_i}{\partial a_k \partial a_l \partial a_m} \right) \right\} \overline{\Delta a_j \Delta a_k \Delta a_l \Delta a_m} \\
&\quad - \frac{1}{4} \sum_{j,k,l,m=1}^{q} \left(\frac{\partial^2 g_i}{\partial a_j \partial a_k} \right) \left(\frac{\partial^2 g_i}{\partial a_l \partial a_m} \right) \overline{\Delta a_j \Delta a_k}\ \overline{\Delta a_l \Delta a_m} + \cdots
\end{aligned} \tag{11}$$

Particularly, when a_i's are statistically independent Gaussian random variables, we obtain the mean and variance, retaining the terms of the fourth-order moments,

$$\overline{g_i(x, a)} = g_i(x, \bar{a}) + \frac{1}{2}\sum_{j=1}^{q}\left(\frac{\partial^2 g_i}{\partial a_j^2}\right)\overline{\Delta a_j^2} + \frac{1}{8}\sum_{j,k=1}^{q}\frac{\partial^4 g_i}{\partial a_j^2 \partial a_k^2}\overline{\Delta a_j^2}\,\overline{\Delta a_k^2} \tag{12}$$

$$\sigma_{g_i}^2 = \sum_{j=1}^{q}\left(\frac{\partial g_i}{\partial a_j}\right)^2\overline{\Delta a_j^2} + \sum_{j,k=1}^{q}\left[\frac{1}{2}\left(\frac{\partial^2 g_i}{\partial a_j \partial a_k}\right)^2 + \left(\frac{\partial g_i}{\partial a_j}\right)\left(\frac{\partial^3 g_i}{\partial a_j \partial a_k^2}\right)\right]\overline{\Delta a_j^2 \Delta a_k^2} \tag{13}$$

Using the mean and the variance thus obtained, we transform the chance-constraint (3) into the deterministic nonlinear constraint:

$$\overline{g_i(x, a)} - \lambda_i \sigma_{g_i} \geq 0 \tag{14}$$

where λ_i is the undetermined multiplier and determined as shown below.

The validity of the transformation (14) is proved in Appendix A by using Tchebychev inequality

$$\text{Prob.}\,[\,|g_i - \bar{g}_i| \geq \lambda_i \sigma_{g_i}] \leq \frac{1}{\lambda_i^2} \tag{15}$$

Further, we can show that the chance-constraint (3) is necessarily satisfied when

$$\lambda_i \geq 1/\sqrt{1-P_i} \tag{16}$$

If we choose $\lambda_i = 1/\sqrt{1-P_i}$ from (16), the optimal solution is sometimes too conservative because the condition (16) is sufficient but not necessary. In such a case, reducing the value of λ_i in the active constraint, i.e., the one on which the optimal solution lies, we must lower the probability level to its lower limit. By this way, the feasible region becomes wider and thus the expected value of the objective function may be improved.

Similarly, the chance-constraint on the objective functions is transformed into

$$(\beta - 1)\bar{z} - \lambda_z \sigma_z \geq 0 \tag{17}$$

where

$$\begin{aligned}\bar{z} &= f(x, \bar{c}) + \frac{1}{2}\sum_{j,k=1}^{r}\frac{\partial^2 f}{\partial c_j \partial c_k}\overline{\Delta c_j \Delta c_k} + \frac{1}{6}\sum_{j,k,l=1}^{r}\frac{\partial^3 f}{\partial c_j \partial c_k \partial c_l}\overline{\Delta c_j \Delta c_k \Delta c_l} \\ &\quad + \frac{1}{24}\sum_{j,k,l,m=1}^{r}\frac{\partial^4 f}{\partial c_j \partial c_k \partial c_l \partial c_m}\overline{\Delta c_j \Delta c_k \Delta c_l \Delta c_m} + \cdots\end{aligned} \tag{18}$$

$$\begin{aligned}\sigma_z^2 &= \sum_{j,k=1}^{r}\left(\frac{\partial f}{\partial c_j}\right)\left(\frac{\partial f}{\partial c_j}\right)\overline{\Delta c_j \Delta c_k} + \sum_{j,k,l=1}^{r}\left(\frac{\partial f}{\partial c_j}\right)\left(\frac{\partial^2 f}{\partial c_k \partial c_l}\right)\overline{\Delta c_j \Delta c_k \Delta c_l} \\ &\quad + \frac{1}{12}\sum_{j,k,l,m=1}^{r}\left[3\left(\frac{\partial^2 f}{\partial c_j \partial c_k}\right)\left(\frac{\partial^2 f}{\partial c_l \partial c_m}\right) + 4\left(\frac{\partial f}{\partial c_j}\right)\left(\frac{\partial^3 f}{\partial c_k \partial c_l \partial c_m}\right)\right]\overline{\Delta c_j \Delta c_k \Delta c_l \Delta c_m} \\ &\quad - \frac{1}{4}\sum_{j,k,l,m=1}^{r}\left(\frac{\partial^2 f}{\partial c_j \partial c_k}\right)\left(\frac{\partial^2 f}{\partial c_l \partial c_m}\right)\overline{\Delta c_j \Delta c_k}\,\overline{\Delta c_l \Delta c_m} + \cdots\end{aligned} \tag{19}$$

and λ_z is the undetermined multiplier and adjusted by the way mentioned above.

Next we consider the simpler case when the constraint and the objective function are the additive sums of the Gaussian random variables:

$$g_i(x, a) = \sum_{j=1}^{s} a_{ij} g_{ij}(x) \tag{20}$$

$$f(x, c) = \sum_{j=1}^{r} c_i f_j(x) \tag{21}$$

where a_{ij}'s and c_i's are Gaussian random variables. Equivalent nonlinear constraints to Eqs. (20) and (21) are given by (see Appendix B)

$$\sum_{j=1}^{s} \bar{a}_{ij} g_{ij}(x) + \gamma(P_i)\{\sum_{j,k=1}^{s} \overline{\Delta a_{ij} \Delta a_{ik}} g_{ij}(x) g_{ik}(x)\}^{1/2} \geq 0 \tag{22}$$

$$(\beta-1) \sum_{j=1}^{r} \bar{c}_j f_j(x) - \gamma(P_z)\{\sum_{j,k=1}^{r} \overline{\Delta c_j \Delta c_k} f_j(x) f_k(x)\}^{1/2} \geq 0 \tag{23}$$

where

$$P_i(t) = \int_{\gamma(P_i)}^{\infty} \phi(t) dt \tag{24}$$

$$\phi(t) = 1/\sqrt{2\pi}\, e^{-t^2/2} \tag{25}$$

Thus $\gamma(P_i)$ is not an undetermined multiplier in this case, but directly determined from the value of P_i.

From the discussion given above, we see that the chance-constraints (3) and (4) are transformed into the equivalent nonlinear constraints. Thus the problems 1 and 2 are reduced to the deterministic nonlinear programming problem.

4. Computational Procedure

We will give a possible computational procedure to solve the problems set up in the

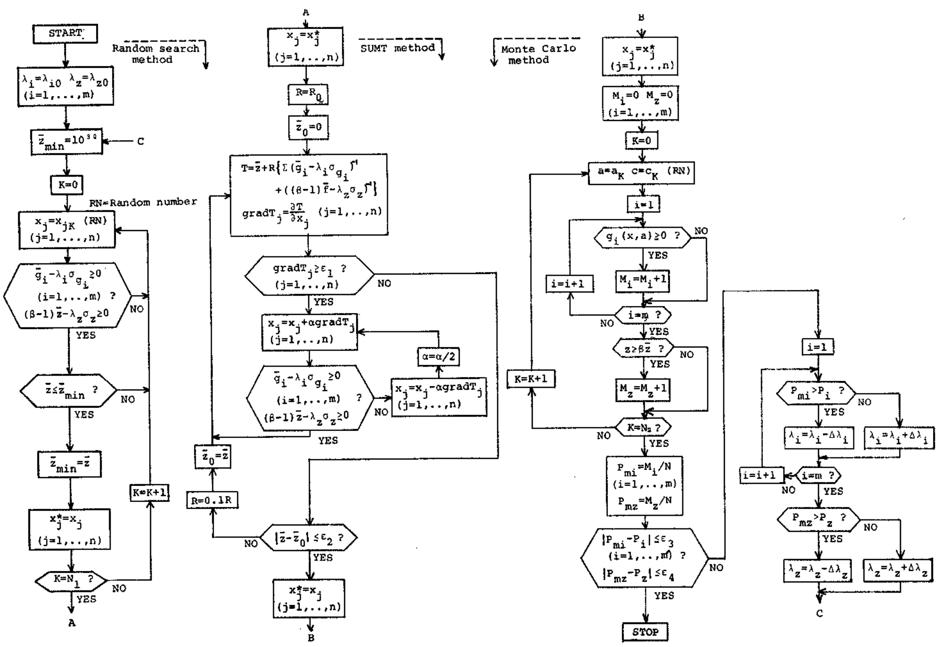

Fig. 1. Computational procedure.

previous section, the flow chart of which is given in Fig. 1. The property of the functions of the nonlinear constraints (14), (17), (22) and (23) are not known in general. Thus, the random search method is employed to make a global search at first. When the local convexity is assured, we can make the solution thus obtained finer by using the techniques of the nonlinear programming. In the figure, we use the SUMT[10] combined with the gradient method, but other methods are also applicable. Further, the calculation of the probability is performed by the Monte Carlo method.

5. Numerical Examples

Let the constraints and the objective function be given by

$$g_1 = a_1 x_1 + a_2 x_2^{a_3} - a_4 \geq 0 \tag{26}$$

$$g_2 = a_5 x_1 + a_6 x_2^2 - a_7 \geq 0 \tag{27}$$

$$x_1,\ x_2 \geq 0 \tag{28}$$

$$z = c_1 x_1^{c_2} + c_3 x_2^2 \tag{29}$$

where the coefficients a_i's and c_i's are statistically independent Gaussian random variables. Let us now consider the following examples:

Example 1 "Specifying the probability levels of satisfying the constraints (26) and (27) to the values higher than P_1 and P_2, minimize the expected value of the objective function (29)."

Example 2 "Specifying the probability level of z's exceeding 1.1 times its optimal value to the value lower than P_z as well as those of the constraints, minimize the expected value of the objective function."

First we consider the chance-constraint on the constraint (26). The mean and the variance are calculated as

$$\bar{g}_1 = \bar{a}_1 x_1 + \bar{a}_2 x_2^{\bar{a}_3} - \bar{a}_4 + \frac{1}{2}\bar{a}_2 x_2^{\bar{a}_3}(\ln x_2)^2 \overline{\Delta a_3^2} + \frac{1}{4}(\ln x_2)^2(\overline{\Delta a_3^2})^2 + \cdots \tag{30}$$

$$\begin{aligned}\sigma_{g_1}^2 = {} & x_1^2 \overline{\Delta a_1^2} + x_2^{2\bar{a}_3}\overline{\Delta a_2^2} + (\ln x_2)^2(\bar{a}_2 x_2^{\bar{a}_3})^2 \overline{\Delta a_3^2} + \overline{\Delta a_4^2} \\ & + 2(\ln x_2)^2 x_2^{2\bar{a}_3}\overline{\Delta a_2^2}\overline{\Delta a_3^2} + \frac{3}{2}\{(\ln x_2)^2 \bar{a}_2 x_2^{\bar{a}_3}\}^2(\overline{\Delta a_3^2})^2 + \cdots\end{aligned} \tag{31}$$

Neglecting the higher order terms "$+\cdots$", we have the nonlinear constraint

$$\begin{aligned}G_1 = {} & \bar{a}_1 x_1 + \bar{a}_2 x_2^{\bar{a}_3} - \bar{a}_4 + \frac{1}{2}\bar{a}_2 x_2^{\bar{a}_3}(\ln x_2)^2\left\{\overline{\Delta a_3^2} + \frac{1}{4}(\ln x_2)^2(\overline{\Delta a_3^2})^2\right\} \\ & - \lambda_1\Big[\overline{\Delta a_1^2}x_1^2 + \overline{\Delta a_2^2}x_2^{2\bar{a}_3} + \overline{\Delta a_3^2}(\bar{a}_2 x_2^{\bar{a}_3})^2(\ln x_2)^2 + \overline{\Delta a_4^2} + 2(\ln x_2)^2 x_2^{2\bar{a}_3}\overline{\Delta a_2^2}\overline{\Delta a_3^2} \\ & + \frac{3}{2}(\overline{\Delta a_3^2})^2\{\bar{a}_2 x_2^{\bar{a}_3}(\ln x_2)^2\}^2\Big]^{1/2} \geq 0\end{aligned} \tag{32}$$

Since Eq. (27) is the additive sum of the Gaussian random variables, we have the constraint equivalent to the chance-constraint

$$G_2 = \bar{a}_5 x_1 + \bar{a}_6 x_2^2 + \gamma(P_2)[\overline{\Delta a_5^2} x_1^2 + \overline{\Delta a_6^2} x_2^4 + \overline{\Delta a_7^2}]^{1/2} - \bar{a}_7 \geq 0 \tag{33}$$

Similarly, the chance-constraint on the objective function is transformed into the nonlinear constraint. The mean and the variance are

$$\bar{z} = \bar{c}_1 x_1^{\bar{c}_2} + \bar{c}_3 x_2^2 + \frac{1}{2} \bar{c}_1 x_1^{\bar{c}_2} (\ln x_1)^2 \left\{ \overline{\Delta c_2^2} + \frac{1}{4} (\ln x_1)^2 (\overline{\Delta c_2^2})^2 + \cdots \right\} \tag{34}$$

$$\sigma_z^2 = \overline{\Delta c_1^2} (x_1^{\bar{c}_2})^2 + \overline{\Delta c_2^2} \{\bar{c}_1 (\ln x_1) x_1^{\bar{c}_2}\}^2 + \overline{\Delta c_3^2} x_2^4 + \frac{3}{2} (\overline{\Delta c_2^2})^2 \{\bar{c}_1 (\ln x_1)^2 x_1^{\bar{c}_2}\}^2$$
$$+ 2(x_1^{\bar{c}_2} \ln x_1)^2 \overline{\Delta c_1^2}\, \overline{\Delta c_2^2} + \cdots \tag{35}$$

Thus, retaining up to the fourth-order moments, we have

$$G_z = (\beta - 1) \left[\bar{c}_1 x_1^{\bar{c}_2} + \bar{c}_3 x_2^2 + \frac{1}{2} \bar{c}_1 x_1^{\bar{c}_2} (\ln x_1)^2 \left\{ \overline{\Delta c_2^2} + \frac{1}{4} (\ln x_1)^2 (\overline{\Delta c_2^2})^2 \right\} \right]$$
$$- \lambda_z \left[\overline{\Delta c_1^2} (x_1^{\bar{c}_2})^2 + \overline{\Delta c_2^2} \{\bar{c}_1 (\ln x_1) x_1^{\bar{c}_2}\}^2 + \overline{\Delta c_3^2} x_2^4 + 2(x_1^{\bar{c}_2} \ln x_1)^2 \overline{\Delta c_1^2}\, \overline{\Delta c_2^2} \right.$$
$$\left. + \frac{3}{2} (\overline{\Delta c_2^2})^2 \{\bar{c}_1 (\ln x_1)^2 x_1^{\bar{c}_2}\}^2 \right]^{1/2} \geq 0 \tag{36}$$

Now, consider the case when the numerical data of the coefficients are given by

$$\begin{aligned} &\bar{a}_1 = 1, \quad \sigma_{a_1} = 0.1, \quad \bar{a}_2 = 1, \quad \sigma_{a_2} = 0.1, \quad \bar{a}_3 = 1, \quad \sigma_{a_3} = 0.1, \quad \bar{a}_4 = 1, \\ &\sigma_{a_4} = 0.1, \quad \bar{a}_5 = 1, \quad \sigma_{a_5} = 0.1, \quad \bar{a}_6 = -1, \quad \sigma_{a_6} = 0.1, \quad \bar{a}_7 = 0, \quad \sigma_{a_7} = 0.1, \\ &\bar{c}_1 = 1, \quad \sigma_{c_1} = 0.1, \quad \bar{c}_2 = 2, \quad \sigma_{c_2} = 0.2, \quad \bar{c}_3 = 2, \quad \sigma_{c_3} = 0.2 \end{aligned} \tag{37}$$

To verify the validity of the computational procedure given in Section 4, first we consider the case when the coefficients are replaced by their mean values, in which the exact solution is easily obtained. Thus the problem becomes as follows: Under the constraints

$$x_1 + x_2 - 1 \geq 0 \tag{38}$$

$$x_1 - x_2^2 \geq 0 \tag{39}$$

$$x_1,\ x_2 \geq 0 \tag{40}$$

minimize the objective function

$$z = x_1^2 + 2x_2^2 \tag{41}$$

The feasible region is shaded in Fig. 2. Since the contours of the objective function is elliptic, the optimal solution lies on the contour tangent to the line $x_1 + x_2 - 1 = 0$, i.e., $x_1^* = 2/3$, $x_2^* = 1/3$. Thus the optimal value of the objective function is $z^* = 2/3$. The solution obtained by applying the authors' algorithm is given in Table 1 and compared with the exact solution, in which the stop command of the computation is given by $\varepsilon_2 = 0.001$. As seen from the table, the solution is reasonable. If we want the more accurate solution, we have only to make the value of ε_2 smaller. The values of P_1 and P_2 in Table 1 are the probabilities that the constraints (26) and (27) are satisfied when we use the solution obtained by replacing the coefficients by their mean values. From

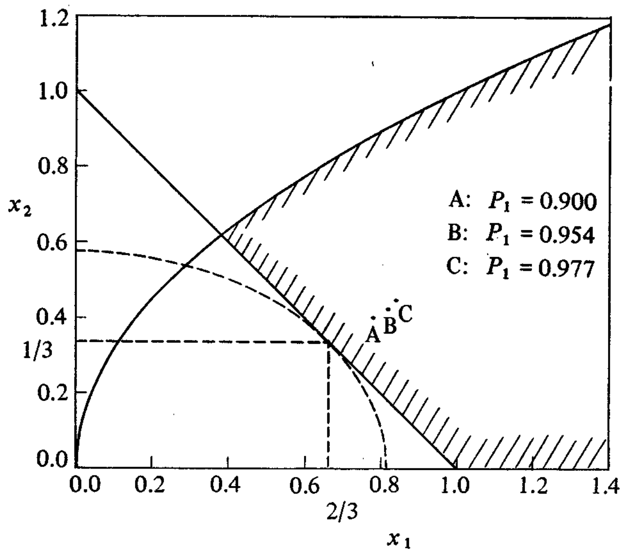

Fig. 2. Feasible region.

Table 1.

	x_1*	x_2*	$\overline{z^*}$	P_1	P_2	P_Z	G_1	G_2	G_Z
Authors' algorithm	0.6665	0.3336	0.667	0.506	1	0.133	8.7×10^{-5}	5.6×10^{-1}	6.7×10^{-2}
Exact solution	2/3	1/3	2/3	—	—	—	—	—	—

this, we find that the active constraint (26) is satisfied with probability 0.5, approximately. Further, P_z is the probability that the objective function exceed 1.1 times its optimal value. It should be noted that the trials of the Monte Carlo simulation is 5000.

Let us now consider Example 1. The calculations are carried out for several values of the undetermined multiplier λ_1 while the probability level of the constraint (27) is specified to 0.95. The mean and the variance are calculated up to the second- and fourth-order moments, respectively. The results are listed in Table 2. The columns G_1, G_2 and G_z are the lists of the values of the transformed functions (32), (33) and (36). As the value of λ_1 increases, the probability level rises. It is clear that the constraint (38) is effective, for the value of G_1 is nearly equal to zero, i.e., the solutions are on the boundary of the constraint (38). The values of probability P_1 are plotted against λ_1 in Fig. 3.

Table 2.

λ_1	λ_Z	x_1*	x_2*	$\overline{z^*}$	P_1	P_2	P_Z	G_1	G_2	G_Z
1	0	0.7478	0.3833	0.859	0.839	1	0.111	8.0×10^{-5}	4.0×10^{-1}	1.4×10^{-2}
1.3	0	0.7763	0.4013	0.925	0.900	1	0.104	7.8×10^{-5}	4.1×10^{-1}	9.3×10^{-2}
1.5	0	0.7957	0.4116	0.973	0.929	1	0.098	7.7×10^{-5}	4.1×10^{-1}	9.7×10^{-2}
1.7	0	0.8128	0.4247	1.022	0.954	1	0.093	7.5×10^{-5}	4.2×10^{-1}	1.0×10^{-1}
2	0	0.8356	0.4480	1.100	0.977	1	0.086	7.3×10^{-5}	4.2×10^{-1}	1.1×10^{-1}

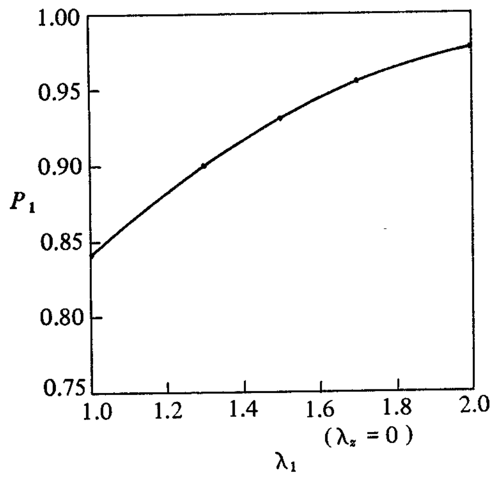

Fig. 3. P_1 against λ_1.

Using such a plot, we can determine the value of the undetermined multiplier for the specified probability level.

Further, to see the transition of the optimal solutions corresponding to the probability levels, we plot them in Fig. 1. As seen from the figure, the solution becomes conservative as the probability level rises.

Next we consider Example 2. The computation results are listed in Tables 3 to 5, where the mean and the variance are calculated up to the second- and the fourth-order moments, respectively. First we discuss the case when λ_1 is fixed to 1.0 and λ_z is made variable. If λ_z exceeds 1.2, the constraint on the objective function begins to be effective, which is known from the fact that the solution approaches to the boundary of

Table 3.

λ_1	λ_z	x_1^*	x_2^*	$\overline{z^*}$	P_1	P_2	P_z	G_1	G_2	G_z
1	0	0.7478	0.3833	0.859	0.839	1	0.111	8.0×10^{-5}	4.0×10^{-1}	1.4×10^{-2}
1	1.2	0.7497	0.3846	0.859	0.839	1	0.111	8.0×10^{-5}	4.0×10^{-1}	1.9×10^{-4}
1	1.3	0.7800	0.4767	1.064	0.967	1	0.092	1.2×10^{-1}	3.4×10^{-1}	5.3×10^{-6}
1	1.35	0.8381	0.5202	1.244	0.992	1	0.082	2.1×10^{-1}	3.5×10^{-1}	4.5×10^{-6}
1	1.4	0.9223	0.6063	1.586	1	1	0.074	3.8×10^{-1}	3.2×10^{-1}	3.0×10^{-6}

G_z (see Table 3). When $\lambda_z = 1.0$, only the chance-constraint on the constraint (38) is effective and thus that on the objective function is not so (Table 4). On the contrary, when $\lambda_z = 1.4$, only the chance-constraint on the objective function is effective (Table 5). In order to determine the value of λ_1 and λ_z for the specified probability level, the technique such as the one used in Example 1 may be resorted to.

Lastly, we discuss the effect of the terms in the calculation of the mean and the variance. The optimal solutions for the two cases are given in Table 6. One is the case when the mean and the variance are calculated up to the second- and the fourth-

Table 4.

λ_1	λ_Z	x_1^*	x_2^*	$\bar{z}^*$	P_1	P_2	P_Z	G_1	G_2	G_Z
1	1	0.7511	0.3833	0.859	0.839	1	0.111	8.0×10^{-5}	4.0×10^{-1}	1.4×10^{-2}
1.4	1	0.7861	0.4063	0.949	0.914	1	0.102	7.7×10^{-5}	4.1×10^{-1}	1.9×10^{-2}
1.6	1	0.8038	0.4185	0.997	0.942	1	0.096	7.5×10^{-5}	4.2×10^{-1}	2.1×10^{-2}
2	1	0.8401	0.4438	1.100	0.977	1	0.083	7.1×10^{-5}	4.3×10^{-1}	3.6×10^{-2}

Table 5.

λ_1	λ_Z	x_1^*	x_2^*	$\bar{z}^*$	P_1	P_2	P_Z	G_1	G_2	G_Z
1	1.4	0.9223	0.6063	1.586	1	1	0.074	3.8×10^{-1}	3.2×10^{-1}	3.0×10^{-6}
1.8	1.4	0.9233	0.6063	1.586	1	1	0.074	2.6×10^{-1}	3.2×10^{-1}	3.0×10^{-6}
2.0	1.4	0.9223	0.6063	1.586	1	1	0.074	2.3×10^{-1}	3.2×10^{-1}	3.0×10^{-6}

Table 6.

	λ_1	λ_Z	x_1^*	x_2^*	$\bar{z}^*$
Mean: up to second order moments Variance: up to fourth order moments	1.6	1	0.8038	0.4185	0.997
Mean: up to zeroth order moments Variance: up to second order moments	1.6	1	0.8013	0.4224	0.999

order moments, and the other is the case when the mean is calculated up to the zeroth-order moment, i.e., the coefficients are replaced by their mean values and the variances up to the second- order moments. The discrepancies are not so great in this problem. Thus the latter is preferable, which is simpler to calculate. The conclusion, however, may not be true in general.

6. Conclusion

This paper is concerned with the formulation and solution of the stochastic nonlinear programming problem by using the chance-constrained concept. The problems are set up to minimize the expected value of the objective function, specifying the probability levels with which the constraints and/or the objective function are to be satisfied. It is shown that they are transformed into the deterministic nonlinear programming problems and an algorithm to systematically solve them is also presented. Further, the numerical examples are provided to illustrate the procedure and its validity.

Although we have not discussed the property of the transformed constraints, it must be studied in the future.

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Appendix A

Theorem: 1) *The supremum of* λ_i *is* $1/\sqrt{1-P_i}$.

2) *The nonlinear constraint* (14) *with* λ_i *satisfying the inequality* (16) *is a sufficient condition for the chance-constraint* (3).

Proof. Tchebychev inequality (15) is rewritten as

$$\text{Prob. } [g_i \geq \bar{g} + \lambda_i \sigma_{g_i} \quad \text{or} \quad g_i \leq \bar{g}_i - \lambda_i \sigma_{g_i}] < 1/\lambda_i^2 \tag{A-1}$$

Thus the inequality

$$\text{Prob. } [g_i \leq \bar{g}_i - \lambda_i \sigma_{g_i}] \leq 1/\lambda_i^2 \tag{A-2}$$

holds. Next we consider the set

$$X_+ = \{x \mid \bar{g}_i - \lambda_i \sigma_{g_i} \geq 0\} \tag{A-3}$$

i.e., the set of x which satisfies the constraint

$$\bar{g}_i - \lambda_i \sigma_{g_i} \geq 0 \tag{A-4}$$

For x contained in the set X_+, the following inequality holds

$$\text{Prob. } [g_i < 0] \leq \text{Prob. } [g_i \leq \bar{g}_i - \lambda_i \sigma_{g_i}] \tag{A-5}$$

From (A-2), we have

$$\text{Prob. } [g_i < 0] \leq 1/\lambda_i^2 \tag{A-6}$$

Using the well known relation:

$$\text{Prob. } [g_i < 0] = 1 - \text{Prob. } [g_i \geq 0] \tag{A-7}$$

we obtain

$$\text{Prob. } [g_i \geq 0] \geq 1 - 1/\lambda_i^2 \tag{A-8}$$

Thus, we get

$$1 - 1/\lambda_i^2 \geq P_i$$

or

$$\lambda_i \geq 1/\sqrt{1-P_i} \tag{A-9}$$

It is clear that the chance-constraint (3) is necessarily satisfied for x which satisfies (A-4) with λ_i satisfying (A-9).

Appendix B

The sum of the Gaussian random variables is also a Gaussian random vraiable. Thus $g_i(x, a)$ and $f(x, c)$ are Gaussian random variables. The means and the variances are given by

$$\overline{g_i(x, a)} = \sum_{j=1}^{s} a_{ij} g_{ij}(x) \tag{B-1}$$

$$\sigma g_i = [\sum_{j,k=1}^{s} \overline{\Delta a_{ij}\Delta a_{ik}}\, g_{ij}(x) g_{ik}(x)]^{1/2} \quad \text{(B-2)}$$

$$\overline{f(x, c)} = \sum_{j=1}^{r} \bar{c}_j f_j(x) \quad \text{(B-3)}$$

$$\sigma_f = [\sum_{j,k=1}^{r} \overline{\Delta c_j \Delta c_k}\, f_j(x) f_k(x)]^{1/2} \quad \text{(B-4)}$$

Thus, the chance-constraint (3) is written as

$$\text{Prob. } [g_i(x, a) \geq 0] = \int_{-\frac{\bar{g}_i}{\sigma g_i}}^{\infty} \phi(t)dt \geq P_i = \int_{\gamma(P_i)}^{\infty} \phi(t)dt \quad \text{(B-5)}$$

which is equivalent to

$$-\bar{g}_i/\sigma g_i \leq r(P_i)$$

or

$$\sum_{j=1}^{s} \bar{a}_{ij} g_{ij}(x) + r(P_i)\{\sum_{j,k=1}^{s} \overline{\Delta a_{ij}\Delta a_{ik}}\, g_{ij}(x) g_{ik}(x)\}^{1/2} \geq 0 \quad \text{(B-6)}$$

Similarly, the constraint (23) is derived.

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