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In a broad sense, all of signal functions can be regarded as the modulated signals. The modulated signal is composed of two independent functions, the carrier signal and the modulating signal, and it can be represented as the section of the modulation model in the three-dimensional space under the consideration of independence between the carrier and the modulating signal.

This paper reports how to obtain one-dimensional Fourier or Laplace transform from the two-dimensional transform which is given by transformation of the modulation model in the three-dimensional space. This method is available for the spectrum analysis, the transient analysis and others.

1. Introduction

Already, several methods have been reported for analizing a modulated signal. Some representatives of them are a method of double Fourier series[1),2)] and a method of multiplex Fourier series. They are inavailable, however, for the non-periodic functions without only case that both a carrier and a modulating signal are periodic functions.

In this paper, it is shown Fourier transform of any signal (or function) is derived from two-dimensional Fourier transform, and any kind of modulated signal can be analized by means of converting two-dimensional Fourier transform to one-dimensional Fourier transform.

In the same manner, the method of converting two-dimensional Laplace transform to the one-dimensional transform. It will be available for the calculation of the transient analysis when a modulated signal is applied to networks as an input signal.

2. Conversion of two-dimensional Fourier transform to one-dimensional transform

The modulated signal $f(t)$ is composed of the carrier signal $g(t)$ and the modulating signal $h(t)$. In general, it is represented as follows;

$$f(t) \equiv f_0(g(t),\ h(t)) \tag{1}$$

The function $f_0(g(t), h(t))$ determines the relation between $g(t)$ and $h(t)$, namely it indicates the manner of modulation.

Fourier transform $F(\omega)$ of $f(t)$ is shown in Eq. (2).

$$F(\omega) = \int_{-\infty}^{\infty} f_0(g(t),\ h(t)) e^{-j\omega t}\, dt \tag{2}$$

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So that

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \tag{3}$$

In general, the calculation of Eq. (2) is difficult and has been made Fourier transformation of the modulated signal troublesome. In order to avoid this difficulty, a two-variable function $f_1(x, y)$, which represents modulation model in three-dimensional space (x, y, z), is introduced.

$$z = f_1(x, y) = f_0(g(x), h(y)) \tag{4}$$

In Eq. (4), the carrier signal $g(x)$ and the modulating signal $h(y)$ are separated, and $g(x)$, $h(y)$ are independent one another, for x, y are independent two variables.

Two-dimensional Fourier transform $F_0(\sigma_1, \sigma_2)$ of the Eq. (4) is given by

$$F_0(\sigma_1, \sigma_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(g(x), h(y))e^{-j(\sigma_1 x+\sigma_2 y)}dx\,dy \tag{5}$$

The calculation of Eq. (5) is easier than that of Eq. (2), for the carrier signal $g(x)$ and the modulating signal $h(y)$ can be treated independently.

Evidently, an inverse transform of Eq. (5) is shown as follows;

$$f_0(g(x), h(y)) = \left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\sigma_1, \sigma_2)e^{j(\sigma_1 x+\sigma_2 y)}d\sigma_1 d\sigma_2 \tag{6}$$

In fact, the modulated signal $f(t)$ is the function of only one variable t, and it appears on the plane $x=y\equiv t$ in the three-dimensional space, for both carrier signal $g(t)$ and modulating signal $h(t)$ are functions of time t. Evidently, the modulated signal $f(t)$ is obtained as a project of a section of the modulating model from $x=y$ plane to $x=0$ or $y=0$ plane in the three-dimensional space.

Putting $x=y=t$ into Eq. (6), it can be rewritten as

$$\begin{aligned} f(t) &= f_0(g(t), h(t)) \\ &= \left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_0(\sigma_1, \sigma_2)e^{j(\sigma_1+\sigma_2)t}d\sigma_1 d\sigma_2 \end{aligned} \tag{7}$$

In Eq. (7), the variable t appears at an exponential term. Inserting Eq. (7) into Eq. (2) and exchanging the order of integration, next Eq. (8) is derived.

$$\begin{aligned} F(\omega) &= \left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_0(\sigma_1, \sigma_2)e^{j(\sigma_1+\sigma_2)t}dt\,d\sigma_1 d\sigma_2 \\ &= \left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_0(\sigma_1, \sigma_2)\cdot 2\pi\delta(\sigma_1+\sigma_2-\omega)d\sigma_1 d\sigma_2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_0(\sigma_1, \omega-\sigma_1)d\sigma_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_0(\omega-\sigma_2, \sigma_2)d\sigma_2 \end{aligned} \tag{8}$$

where $\delta(\sigma)$ indicates a unit impulse function, and it is assumed that the existence of two-dimensional Fourier transform $F_0(\sigma_1, \sigma_2)$ is beforehand ensured. Fourier transform of Eq. (8) is evidently the same Fourier transform in Eq. (1). Eq. (8) proved the fundamental equation for converting two-dimensional Fourier transform into one-dimensional one, and this calculation is comparablly easy.

In this manner, difficulties of the direct calculation of Eq. (2) can be avoided by way of two-dimensional Fourier transform.

3. Some examples

In this section, some of examples are introduced and propriety of this method is shown. At first, an amplitude modulated signal, which contains $\cos \omega_0 t$ as the carrier signal and $h(t)$ as the modulating signal, is analized. And the analysis of PWM signal, whose leading edge is modulated by the modulating signal $h(t)$, is presented. In above two examples, the carrier signal is periodic. For the last example, non-periodic signals are chosen.

Example-1. Amplitude modulated signal

The amplitude (balanced) modulated signal $f(t)$ is represented as the product of the carrier signal $\cos \omega_0 t$ and the modulating signal $h(t)$. The modulation model $z=f_0(g(x), h(y))$ in the three-dimensional space is shown as follows;

$$z = f_0(g(x), h(y)) = h(y) \cdot \cos \omega_0 x \tag{9}$$

Using Eq. (5), two-dimensional Fourier transform $F_0(\sigma_1, \sigma_2)$ is obtained in the next equation.

$$F_0(\sigma_1, \sigma_2) = \pi \{\delta(\sigma_1 - \omega_0) + \delta(\sigma_1 + \omega_0)\} H(\sigma_2) \tag{10}$$

where $H(\sigma_2)$ indicates one-dimensional Fourier transform of $h(y)$. Applying Eq. (8) to Eq. (10), one-dimensional Fourier ttansfoım $F(\omega)$ is given by

$$F(\omega) = \frac{1}{2} \{H(\omega + \omega_0) + H(\omega - \omega_0)\} \tag{11}$$

The above result is well-known and the propriety of this method is proved.

Example-2. Pulse width modulated signal

In Fig. 1, the modulation model of pulse width modulated signal is shown. The rectangular wave, whose leading edge is modulated with the modulation signal $h(t)$ ($\geqq 0$), is used as the carrier signal $g(t)$.

In three-dimensional space, the modulation model $z=f_0(g(x), h(y))$ is represented in next Eq. (12).

$$z = f_0(g(x), h(y)) = \begin{cases} 1, & (kT_0 - Ph(y) \leqq x \leqq kT_0) \\ 0, & (\overline{k-1}T_0 < x < kT_0 - Ph(y)) \\ & k = \cdots, -1, 0, 1 \cdots \end{cases} \tag{12}$$

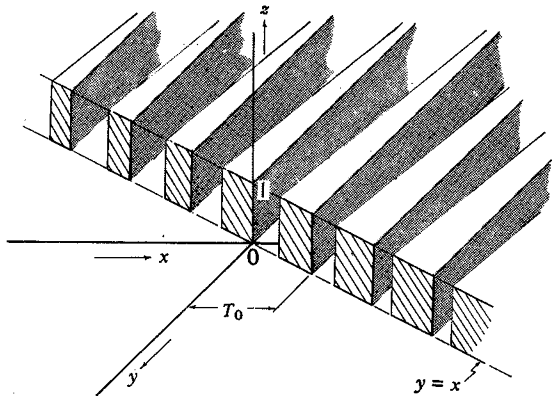

Fig. 1. PWM model in three-dimensional space.

where T_0 indicates the primary interval of the carrier rectangular signal. P is an arbitrary constant which satisfies $0 \leqq \mathrm{P}h(y) \leqq T_0$ and shows a degree of modulation.

The modulation model $f_0(g(x), h(y))$ is the periodic function with regard to the variable x, and it can be expanded to the Fourier series.

$$f(g(x), h(y)) = \frac{\mathrm{P}}{T_0}h(y) + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} j\frac{T_0}{2\pi m}[1 - e^{j(2\pi/T_0)m\mathrm{P}h(y)}]e^{j(2\pi/T_0)mx}$$

$$= \frac{\mathrm{P}}{T_0}h(y) + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} j\frac{T_0}{2\pi m}\left[1 - \sum_{n=0}^{\infty}\left\{j2m\pi\frac{\mathrm{P}}{T_0}h(y)\right\}^n \Big/ n!\right]e^{j(2\pi/T_0)mx} \tag{13}$$

where the exponential term containing the modulation signal $h(y)$ was expanded to Taylor's series. Two-dimensional Fourier transform $F_0(\sigma_1, \sigma_2)$ of $f_0(g(x), h(y))$ is given as follows;

$$F_0(\sigma_1, \sigma_2) = \frac{\mathrm{P}}{T_0}\cdot H(\sigma_2)\cdot 2\pi\delta(\sigma_1) + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} j\frac{T_0}{2\pi m}\left[2\pi\delta(\sigma_2)\right.$$

$$\left. - \sum_{m=0}^{\infty}\left\{\left(j2\pi m\frac{\mathrm{P}}{T_0}\right)^n \cdot H^{n*}(\sigma_2)\right\}\Big/ n!\right]2\pi\delta(\sigma_1 - 2\pi m/T_0) \tag{14}$$

where,

$$H^{n*}(\sigma_2) = \int_{-\infty}^{\infty}\{h(y)\}^n e^{-j\sigma_2 y}dy = \overbrace{H(\sigma_2)*\cdots*H(\sigma_2)}^{n} \tag{15}$$

Applying Eq. (8) to Eq. (14), one-dimensional Fourier transform can be obtained.

$$F(\omega) = \frac{\mathrm{P}}{T_0}H(\omega) + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} j\frac{T_0}{2\pi m}[2\pi\delta(\omega - 2\pi m/T_0) - \sum_{n=0}^{\infty}\{(j2\pi m\mathrm{P}/T_0)^n$$

$$\cdot H^{n*}(\omega - 2\pi m/T_0)\}/n!] \tag{16}$$

In this way, the modulation products in the vicinities of harmonics of the carrier rectangular signal can be calculated.

Example-3. For non-periodic signals

The modulated signal $f(t)=t^2$, which is non-periodic function, is considered. In this case, the modulation model $f_0(g(x), h(y))$ in the three-dimensional space can be chosen as

$$z = f_0(g(x), h(y)) = x \cdot y \tag{17}$$

On the other hand, Fourier transform of t is known as shown in the next equation.

$$\int_{-\infty}^{\infty} t \cdot e^{-j\omega t}\, dt = -j2\pi \frac{d}{d\omega}\delta(\omega) \tag{18}$$

By referring to Eq. (18), two-dimensional Fourier transform of Eq. (17) can be obtained.

$$F_0(\sigma_1, \sigma_2) = \left\{-j2\pi \frac{d}{d\sigma_1}\delta(\sigma_1)\right\}\left\{-j2\pi \frac{d}{d\sigma_2}\delta(\sigma_2)\right\} \tag{19}$$

Applying Eq. (8) to Eq. (19), one-dimensional Fourier transform of Eq. (17) can be given as follows;

$$\begin{aligned} F(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{-j2\pi \frac{d}{d\sigma_1}\delta(\sigma_1)\right\}\left\{-j2\pi \frac{d}{d(\omega-\sigma_1)}\delta(\omega-\sigma_1)\right\}d\sigma_1 \\ &= -2\pi\int_{-\infty}^{\infty}\frac{d}{d\sigma_1}\delta(\sigma_1)\left\{\frac{d}{d(\omega-\sigma_1)}\delta(\omega-\sigma_1)\right\}d\sigma_1 \\ &= -2\pi\,\frac{d^2}{d(\omega-\sigma_1)^2}\delta(\omega-\sigma_1)\Big|_{\sigma_1=0} \\ &= -2\pi\,\frac{d^2}{d\omega^2}\delta(\omega) \end{aligned} \tag{20}$$

This result coincides with that of already known, and the propriety of this method was proved for the non-periodic function.

By this method, Fourier transform of some functions are more easily obtained than the direct method.

4. Conversion from two-dimensional Laplace transform to one-dimensional transform

It is more convenient to use Laplace transform than Fourier transform in some cases, such as the calculation of the transient response of the system, etc. In this section, the method of the conversion of two-dimensional Laplace transform to one-dimensional transform is considered.

The modulated signal $f(t)$ is given as in Eq. (1).

$$f(t) \equiv f_0(g(t), h(t)), \qquad (t \geqq 0) \tag{21}$$

where $g(t)$ and $h(t)$ represent the carrier signal and the modulating signal respectively. Laplace transform $F(s)$ of $f(t)$ is given by the next equation.

$$F(s) = \int_0^{\infty} f_0(g(t), h(t)) \cdot e^{-st}\, dt \tag{22}$$

Evidently, the inverse transform of $F(s)$ in Eq. (22) is given by

$$f(t)=f_0(g(t),\,h(t))=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}F(s)e^{st}\,ds \tag{23}$$

In calculating the Eq. (22), two-dimensional Laplace transform of two-variable function $f_1(x, y)$ of the modulation model in the three-dimensional space (x, y, z), as was presented, is treated. The modulation model is given by

$$z=f_1(x,\,y)=f_0(g(x),\,h(y)) \tag{24}$$

Two-dimensional Laplace transform $F_0(s_1, s_2)$ of Eq. (24) is represented as

$$F_0(s_1,\,s_2)=\int_0^\infty\int_0^\infty f_0(g(x),\,h(y))e^{-s_1x-s_2y}\,dx\,dy \tag{25}$$

The inverse transform of Eq. (25) is given by

$$f_0(g(x),\,h(y))=\left(\frac{1}{2\pi j}\right)^2\int_{c_1-j\infty}^{c_1+j\infty}\int_{c_2-j\infty}^{c_2+j\infty}F_0(s_1,\,s_2)e^{s_1x+s_2y}\,ds_1\,ds_2 \tag{26}$$

Putting $x=y=t$ into the Eq. (26), it represents the same function as Eq. (21) as follows;

$$\begin{aligned}f(t)&=f_0(g(t),\,h(t))\\&=\left(\frac{1}{2\pi j}\right)^2\int_{c_1-j\infty}^{c_1+j\infty}\int_{c_2-j\infty}^{c_2+j\infty}F_0(s_1,\,s_2)\,e^{(s_1+s_2)t}\,ds_1\,ds_2\end{aligned} \tag{27}$$

Only the exponential term in Eq. (27) contains the variable t. Inserting Eq. (27) into Eq. (22) and exchanging the order of integration, one-dimensional Laplace transform is given by

$$\begin{aligned}F(s)&=\left(\frac{1}{2\pi j}\right)^2\int_{c_1-j\infty}^{c_1+j\infty}\int_{c_2-j\infty}^{c_2+j\infty}F_0(s_1,\,s_2)\int_0^\infty e^{(s_1+s_2-s)t}\,dt\,ds_1\,ds_2\\&=\left(\frac{1}{2\pi j}\right)^2\int_{c_1-j\infty}^{c_1+j\infty}\int_{c_2+j\infty}^{c_2+j\infty}F_0(s_1,\,s_2)\cdot\frac{1}{s-s_1-s_2}\,ds_1\,ds_2\\&=\frac{1}{2\pi j}\int_{c_1-j\infty}^{c_1+j\infty}F_0(s_1,\,s-s_1)ds_1=\frac{1}{2\pi j}\int_{c_2-j\infty}^{c_2+j\infty}F_0(s-s_2,\,s_2)ds_2\end{aligned} \tag{28}$$

where, it is assumed that the modulation model $f_0(g(x), h(y))$ is reduced to zero in the range of $x<0$ or $y<0$ for ensuring the existence of $F_0(s_1, s_2)$, and the real part of complex number s_1+s_2-s is negative.

Equation (28) proved the fundamental equation for converting two-dimensional Laplace transform to the one-dimensional transform. The calculation of Eq. (28) can be performed by using the calculus of residues in genenal.

Example; The same example in Eq. (9) is considered again.

$$z=f_0(g(x),\,h(y))=h(y)\cdot\cos\omega_0x\,,\qquad(x\geqq0,\ y\geqq0) \tag{29}$$

Two-dimensional transform $F_0(s_1, s_2)$ is represented as

$$F_0(s_1, s_2) = H(s_2)\cdot\frac{s_1}{s_1^2+\omega_0^2} \tag{30}$$

where $H(s_2)$ indicates Laplace transform of $h(y)$. Applying Eq. (28) to Eq. (30), one-dimensional Laplace transform $F(s)$ is given as follows;

$$\begin{aligned} F(s) &= \frac{1}{2\pi j}\int_{c_1-j\infty}^{c_1+j\infty} H(s-s_1)\cdot\frac{s_1}{s_1^2+\omega_0^2}ds_1 \\ &= \frac{1}{2}\{H(s+j\omega_0)+H(s-j\omega_0)\} \end{aligned} \tag{31}$$

It is evident that this method contains the formula of convolusion. Laplace transform of some kinds of functions can be more easily derived by this method than by the direct one.

5. Conclusions

The modulated signal is originally the function of one variable time t. It contains, two independent functions. One of them is the function of carrier signal and the other is that of modulating signal. The modulation manner of the modulated signal can be easily studied by treating these two functions of t as separate ones independently.

In this paper, it is mentioned how to obtain one-dimensional Fourier or Lalpace transform of the modulated signal from the two-dimensional transform which is given by transformation of the modulation model in the three-dimensional space under the consideration of independence between the functions of the carrier signal and the modulating signal.

In this method, Fourier or Laplace transform can be easily obtained even if the carrier and the modulating signal are non-periodic functions.

It is shown in examples this method contains the convolusion formula in Fourier and Laplace transform when the modulated signal is represented by the product of functions of the carrier and the modulating signals. All of functions can be regarded as modulated signals, and Eq. (8) and Eq. (28) are the generalized convolusion formula in Fourier and Laplace transform respectively. The converted Fourier transform and the converted Laplace transform are adapted for the spectrum anaysis and the transient analysis respectively.

References

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