Four Series of Graphs Derived from a Full Graph

Four Series of Graphs Derived from a Full Graph

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Four series of graphs derived from a full graph by removing four different sets of branches are reported. These four series are named *generalized r-, generalized p-, generalized m-* and *generalized s-Series,* respectively, from the fact that they include four basic series which are called *r-, p-, m-* and *s-Series* in incomplete graphs, as special cases.

The process of reducing the tree determinants made on the basis of the graphs belonging to these four series are described, and then the general formulas for the total number of trees are derived. Furthermore, it is shown that the principal parts of the general formulas in *generalized m-* and *generalized s-Series* can be written down directly from two specific diagrams which are made in accordance with the conditions of given graphs belonging to these series.

1. Introduction

A full graph[1] is defined as an interconnection of all pairs of nodes by $n\,(\geq 1)$ and only n branches. When $n=1$ the full graph reverts to the well-known complete graph. An incomplete graph[2] is derived by removing some branches from a complete graph having the same number of nodes as the incomplete graph, and classing similar kinds of incomplete graphs into series can be done by the form of the branches to be removed from a complete graph. With this method, the following four series then can be considered as most fundamental:

(1) *r-Series:* a series of incomplete graphs derived by removing r branches from different pair of nodes[2].

(2) *p-Series:* a series of incomplete graphs derived by removing p branches from any single node[2].

(3) *m-Series:* a series of incomplete graphs derived by removing m branches which form a loop[3].

(4) *s-Series:* a series of incomplete graphs derived by removing s branches which are connected in series but do not form a loop[3].

In the present paper, these series are called four basic series of incomplete graphs.

If a complete graph, which is used as an original graph in order to derive various incomplete graphs, is replaced with a full graph, it is expected that more generalized series of graphs may be derived from the full graph by removing the specific forms of branches.

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Each of the four series, which is discussed in this paper, contains the same kind of many graphs derived from a full graph according to such a process, and the four basic series of incomplete graphs stated above can be derived directly from these four series.

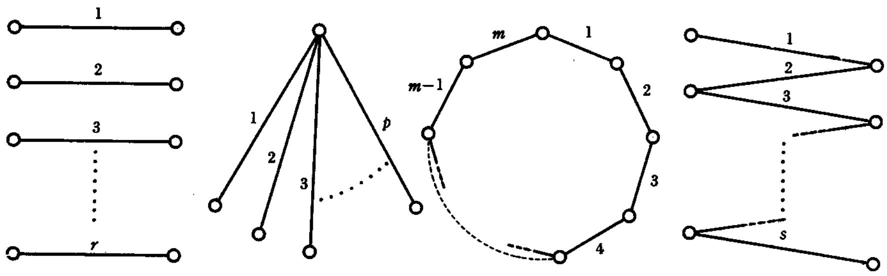
The general formula for the total number of trees in a given graph is obtained by reducing the tree determinant[4] made on the basis of a general diagram of that graph. First the forms of branches to be removed from a full graph are shown and then the process of reducing the tree determinants in the graphs belonging to the four series are described.

The four series in the present paper may be regarded as a generalization of the four basic series of incomplete graphs. In this sence, the authors gave these series the names of *generalized r-*, *generalized p-*, *generalized m-* and *generalized s-Series* respectively.

In general formulas for the total number of trees in the graphs belonging to *generalized m-* and *generalized s-Series*, the principal parts of these formulas can be written down directly from two specific diagrams which are described later. Finally, some examples of the graphs belonging to each of the four generalized series are shown and the total number of their trees is calculated.

2. Sets of branches to be removed from a full graph

In order to clarify the configurations of the graphs which are discussed in this paper, first the forms of branches to be removed from a full graph in the graphs belonging to four generalized series are shown.



(a) *r−Series* (b) *p−Series* (c) *m− Series* (d) *s−Series*

Fig. 1 Forms of branches to be removed from an original complete graph.

Fig.1 shows the forms of branches to be removed from an original complete graph in the graphs belonging to four basic series of incomplete graphs. In these figures, evidently there is but one branch in some pairs of the nodes.

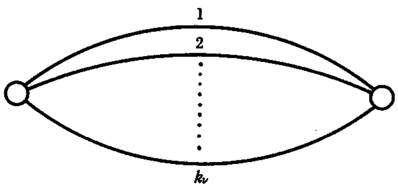


Fig. 2 Unit of branches.

We now consider the unit of branches, which has k_ν branches in parallel between two nodes, such as shown in Fig.2. If all the branches in Fig.1 are replaced with required numbers of this unit, they become as shown in Fig.3. Such groups of branches are sets of branches to be removed from a full graph in order to obtain the graphs belonging to four series under consideration.

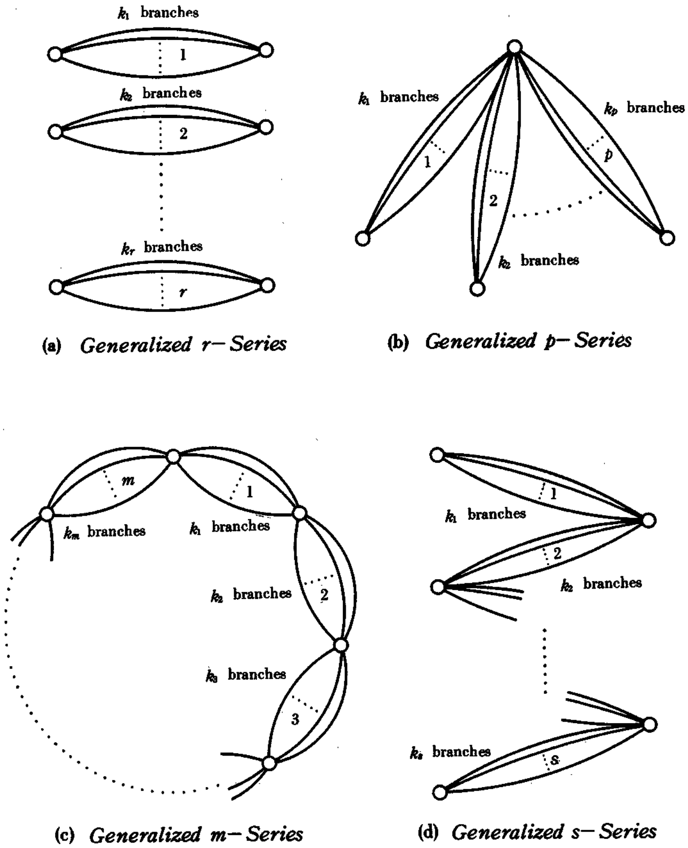


Fig. 3 Forms of branches to be removed from a full graph in the graphs belonging to four series under consideration.

3. Process of reducing the tree determinants and general formulas for the total number of trees

The total number of trees, T, in a given graph of N nodes can be found in principle by calculating the following tree determinant:[4)]

$$T=\begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1a} \\ t_{21} & t_{22} & \cdots & t_{2a} \\ \vdots & \vdots & & \vdots \\ t_{a1} & t_{a2} & \cdots & t_{aa} \end{vmatrix}, \tag{1}$$

where $\alpha = N-1$, t_{ii} represents the number of branches that touch node i and $t_{ij}(i \neq j)$ represents the negative of the number of branches that join node i to node j.

The tree determinants of the symmetrical matrix, which give the total number of trees in the graphs belonging to the four series under consideration, are made on the basis of the general diagrams of the graphs derived from a full graph by removing the sets of branches shown in Fig.3. In this chapter, the process of reducing the tree determinants and the general formulas for the total number of trees obtained as the results are shown.

3.1 *Generalized r–Series*

Making the determinant representing the total number of trees in the graphs belonging to this series, we can find that the elements in Eq.(1) become

$$\left.\begin{array}{l} t_{2\mu-1,2\mu-1} = t_{2\mu,2\mu} = n(N-1) - k_\mu, \\ t_{2\mu-1,2\mu} = t_{2\mu,2\mu-1} = -(n-k_\mu), \quad (\mu = 1, 2, \cdots, r), \\ t_{qq} = n(N-1), \quad (q = 2r+1, 2r+2, \cdots, N-1), \\ \text{and the other elements} = -n \end{array}\right\} \quad (2)$$

In this determinant of the symmetrical matrix, by adding all elements of all other rows except the N-1th row to the corresponding elements of the N-1th row, and performing similar operation to the first and the following rows except the N-1th row, namely, by adding the elements of the N-1th row to all corresponding elements of the rows just mentioned above, we can put a factor $n(nN)^{N-2-2r}$ out of the determinant. In this operation, row and column can be interchanged. Then by adding the elements of each even row in the remaining determinant of the $2r$ order square matrix to the corresponding elements of each odd row, which is just before the even row, and by subtracting the elements of each odd column from the corresponding elements of each even column, which follows just after the odd column, the tree determinant having the elements of Eq.(2) is reduced as follows:

$$T = n(nN)^{N-2-r} \prod_{i=1}^{r} (nN - 2k_i),$$
$$2 \leqq 2r \leqq N, \qquad 0 \leqq k_i \leqq n. \quad (3)$$

Eq.(3) gives the general formula for the total number of trees in the graphs belonging to *generalized r-Series.*

3.2 *Generalized p–Series*

In the graphs belonging to this series, the value of each element obtained by making the tree determinant of Eq.(1) becomes as follows:

$$\left.\begin{array}{ll} t_{qq} = \begin{cases} n(N-1) - \sum_{i=1}^{p} k_i, & (q=1), \\ n(N-1) - k_{q-1}, & (2 \leqq q \leqq p+1), \\ n(N-1), & (p+2 \leqq q \leqq N-1), \end{cases} & \\ t_{q1} = t_{1q} = -(n - k_{q-1}), \quad (2 \leqq q \leqq p+1), & \\ \text{and the other elements} = -n. & \end{array}\right\} \quad (4)$$

In the first step of operations for reducing the determinant having the elements of Eq.(4), by performing the same procedure as the method used in *generalized r-Series,* a factor $n(nN)^{N-3-p}$ is put out of the original determinant and the remaining determinant of square matrix has the order of $p+1$. Then by the ordinary method the remaining determinant is expanded. Thus, the total number of trees in the graphs belonging to this series is given by

$$T=n(nN)^{N-3-p}\Big[(nN-\sum_{i=1}^{p}k_i)\prod_{i=1}^{p}(nN-k_i) -\sum_{i=1}^{p}k_i^2\prod_{j=1}^{p}\{(1-\delta_{ij})(nN-k_j)\}\Big],$$
$$1\leqq p\leqq N-1,\qquad 0\leqq k\leqq n, \tag{5}$$

where δ_{ij} is Kronecker's delta, that is,

$$\delta_{ij}=\begin{cases}1 & \text{for } i=j\,,\\ 0 & \text{for } i\neq j\,.\end{cases} \tag{6}$$

3.3 *Generalized m–Series*

The elements of the tree determinant made from a general diagram of graphs belonging to this series have the following values:

$$\left.\begin{array}{ll} t_{qq}=\begin{cases} n(N-1)-(k_m+k_1), & (q=1),\\ n(N-1)-(k_{q-1}+k_q), & (2\leqq q\leqq m),\\ n(N-1), & (m+1\leqq q\leqq N-1),\end{cases} & \\ t_{q,q-1}=t_{q-1,q}=-(n-k_{q-1}), & (2\leqq q\leqq m),\\ t_{1m}=t_{m1}=-(n-k_m), & \\ \text{and the other elements} = -n. & \end{array}\right\} \tag{7}$$

Reducing the determinant of the symmetrical matrix having the elements of Eq.(7) by performing the same procedure as the method used in *generalized r-Series,* we can put a factor $n(nN)^{N-2-m}$ out of the original determinant. The remaining determinant of symmetrical matrix, which is denoted by D_m, has the order of m and becomes as follows:

$$D_m=\begin{vmatrix} nN-(k_m+k_1) & k_1 & 0 & \cdots & 0 & k_m\\ k_1 & nN-(k_1+k_2) & k_2 & \ddots & & 0\\ 0 & k_2 & nN-(k_2+k_3) & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ 0 & & \ddots & \ddots & \ddots & k_{m-1}\\ k_m & 0 & \cdots & 0 & k_{m-1} & nN-(k_{m-1}+k_m) \end{vmatrix}. \tag{8}$$

Let $nN-(k_{q-1}+k_q)$, ($q=1,2,\cdots,m$), in Eq.(8) be $D_{q-1,q}$, where $D_{01}=D_{m1}$. If Eq.(8) is expanded for some simple values of m and their expanded equations are expressed with the aid of the notation just mentioned above, we obtain the following equations:

$$\left.\begin{aligned}
D_3 &= D_{12} D_{23} D_{31} - P_{13} + 2k_1 k_2 k_3 , \\
D_4 &= D_{12} D_{23} D_{34} D_{41} - P_{14} + P_{24} - 2k_1 k_2 k_3 k_4 , \\
D_5 &= D_{12} D_{23} D_{34} D_{45} D_{51} - P_{15} + P_{25} + 2k_1 k_2 k_3 k_4 k_5 , \\
D_6 &= D_{12} D_{23} D_{34} D_{45} D_{56} D_{61} - P_{16} + P_{26} - P_{36} - 2k_1 k_2 k_3 k_4 k_5 k_6 , \\
&\cdots\cdots ,
\end{aligned}\right\} \quad (9)$$

where

$$\left.\begin{aligned}
P_{13} &= k_1^2 D_{23} + k_2^2 D_{31} + k_3^2 D_{12} , \\
P_{14} &= k_1^2 D_{23} D_{34} + k_2^2 D_{34} D_{41} + k_3^2 D_{41} D_{12} + k_4^2 D_{12} D_{23} , \\
P_{15} &= k_1^2 D_{23} D_{34} D_{45} + k_2^2 D_{34} D_{45} D_{51} + k_3^2 D_{45} D_{51} D_{12} + k_4^2 D_{51} D_{12} D_{23} + k_5^2 D_{12} D_{23} D_{34} , \\
P_{16} &= k_1^2 D_{23} D_{34} D_{45} D_{56} + k_2^2 D_{34} D_{45} D_{56} D_{61} + k_3^2 D_{45} D_{56} D_{61} D_{12} + k_4^2 D_{56} D_{61} D_{12} D_{23} \\
&\quad + k_5^2 D_{61} D_{12} D_{23} D_{34} + k_6^2 D_{12} D_{23} D_{34} D_{45} , \\
&\cdots\cdots , \\
P_{24} &= k_1^2 k_3^2 + k_2^2 k_4^2 , \\
P_{25} &= k_1^2 k_3^2 D_{45} + k_1^2 k_4^2 D_{23} + k_2^2 k_4^2 D_{51} + k_2^2 k_5^2 D_{34} + k_3^2 k_5^2 D_{12} , \\
P_{26} &= k_1^2 k_3^2 D_{45} D_{56} + k_1^2 k_4^2 D_{23} D_{56} + k_1^2 k_5^2 D_{23} D_{34} + k_2^2 k_4^2 D_{56} D_{61} + k_2^2 k_5^2 D_{34} D_{61} + k_2^2 k_6^2 D_{34} D_{45} \\
&\quad + k_3^2 k_5^2 D_{61} D_{12} + k_3^2 k_6^2 D_{12} D_{45} + k_4^2 k_6^2 D_{12} D_{23} , \\
&\cdots\cdots , \\
P_{36} &= k_1^2 k_3^2 k_5^2 + k_2^2 k_4^2 k_6^2 , \\
&\cdots\cdots .
\end{aligned}\right\} \quad (10)$$

In order to give an unificative interpretation to Eq.(10), we now consider a specific diagram such as shown in Fig.4. This figure represents a convex polygon having

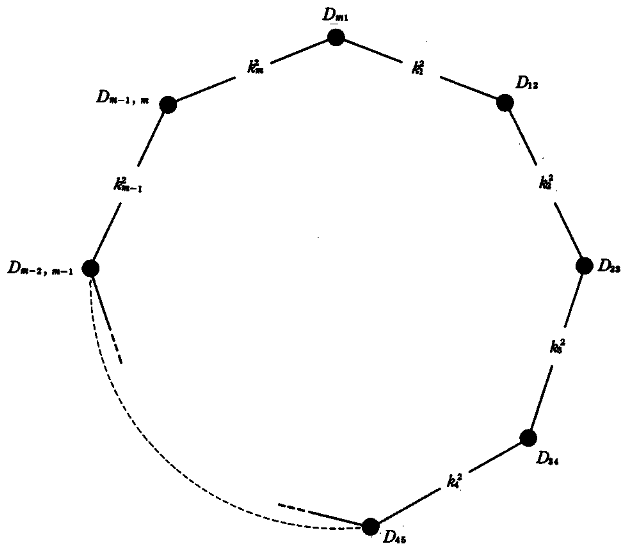


Fig. 4 Diagram in explanation of polynomial P_{im}.

the same number of vertics and edges as the value of m, and the values ot $D_{q-1,q}$ are written close by the vertics in order and the values of k_1^2, k_2^2, — , k_m^2 are written on the edges in order.

We now represent the left sides of Eq.(10) by a symbol P_{im} . Thus, P_{im} is given as the sum of the products of values written on unadjoining edges and values written close by the vertics except the endpoints of their edges, and the right sides of Eq.(10) can be written down directly from a specific diagram shown in Fig.4 by using this procedure.

From Eqs.(9) and (10), and the above consideration, the determinant of symmetrical matrix of the m order, shown in Eq.(8), is expanded as follows:

$$D_m=\prod_{q=1}^{m} D_{q-1,q}+\sum_{i=1}^{l_m} (-1)^i P_{im}+(-1)^{m-1}\, 2 \prod_{j=1}^{m} k_j, \tag{11}$$

where l_m depends on m and is given by

$$l_m=\begin{cases}(m-1)/2\,, & (m \text{ odd}),\\ m/2\,, & (m \text{ even}).\end{cases} \tag{12}$$

From Eq.(11), finally we obtain the total number of trees in the graphs belonging to *generalized m-Series* as

$$T=n(nN)^{N-2-m}\{\prod_{q=1}^{m} D_{q-1,q}+\sum_{i=1}^{l_m} (-1)^i P_{im}+(-1)^{m-1}\, 2 \prod_{j=1}^{m} k_j\},$$
$$3 \leqq m \leqq N\,, \qquad 0 \leqq k_j \leqq n. \tag{13}$$

3.4 *Generalized s–Series*

By comparing the form of branches, shown in Fig.3(d), to be removed from a full graph in the graphs belonging to this series with that, shown in Fig.3(c), to be removed from a full graph in the graphs belonging to *generalized m-Series,* it can be seen that the former is derived from the latter as a special case, that is, the former is obtained from the latter by putting $k_m=0$ and by rewriting m-1 to s. Consequently, the general formula for the total number of trees in the graphs belonging to *generalized s-Series* can be derived directly from Eq.(13) by performing the operations stated above.

We thus obtain the following formula:

$$T=n(nN)^{N-3-s}[\prod_{q=1}^{s+1} D_{q-1,q}+\sum_{i=1}^{l_s} (-1)^i P_{i,s+1}], \qquad 2 \leqq s \leqq N-1\,, \tag{14}$$

where l_s depends on s and is given by

$$l_s=\begin{cases}(s+1)/2\,, & (s \text{ odd}),\\ s/2\,, & (s \text{ even}),\end{cases} \tag{15}$$

and $P_{i,s+1}$ is a polynomial of N, which is obtained from P_{im} in Eq.(10) by using the following relations:

$$\left.\begin{aligned} & m-1=s\,,\\ & k_m=k_{s+1}=0,\ \ k_{m-1}=k_s\,,\ \ k_{m-2}=k_{s-1}\,,\ \text{----}\,,\\ & D_{m1}=nN-(k_m+k_1)=D_{s+1,1}=nN-(k_{s+1}+k_1)=nN-k_1\equiv D_1\,,\\ & D_{m-1,m}=nN-(k_{m-1}+k_m)=D_{s,s+1}=nN-(k_s+k_{s+1})=nN-k_s\equiv D_s.\end{aligned}\right\} \tag{16}$$

By introducing the relations of Eq.(16) into Fig.4, it can be rewritten as shown in Fig.5. From this figure, $P_{i,s+1}$ in Eq.(14) is given as the sum of the products of values

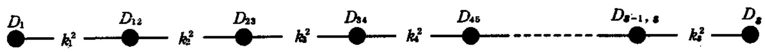


Fig. 5 Diagram in explanation of polynomial $P_{i,\,s+1}$.

written on i unadjoining sections and values written close by the points except the endpoints of their sections.

Furthermore, if it is assumed that $k_0=0$, Eq.(14) can be also applied for $s=1$.

4. Some specific series derived from four generalized series

By giving some conditions to the number of branches to be removed from an original full graph, some specific series are derived from the four generalized series discussed in the previous chapter, as special cases. In this chapter, such the series are considered and the general formulas for the total number of trees in the graphs belonging to these series are given. Furthermore, it is clarified that the four generalized series include the four basic series of incomplete graphs.

4.1 Specific series derived from *generalized r-Series*

It is obvious that Formula (3) is simplified as the following equation in the case of $k_1=k_2=\text{----}=k_r\equiv k$:

$$T=n(nN-2k)^r\,(nN)^{N-2-r}, \qquad 0\leqq k\leqq n. \tag{17}$$

Let $k=n$. In this case, Formula (17) becomes

$$T=n^{N-1}\,(N-2)^r\,N^{N-2-r}. \tag{18}$$

Putting $k=n=1$ in Formula (17),

$$T=(N-2)^r\,N^{N-2-r} \tag{19}$$

is obtained. Formula (19) gives the total number of trees in incomplete graphs belonging to *r-Series*[2], one of the basic series in incomplete graphs derived from a complete graph.

4.2 Specific series derived from *generalized p-Series*

In the case of $k_1=k_2=\text{----}=k_p\equiv k$, the following equation is obtained from Formula (5), that is,

$$T=n(nN-k)^{p-1}\{nN-(1+p)k\}(nN)^{N-2-p}, \quad 0\leqq k\leqq n. \tag{20}$$

Let $k=n$, then from Formula (20), we have

$$T=n^{N-1}(N-1)^{p-1}(N-1-p)N^{N-2-p}. \tag{21}$$

Putting $k=n=1$ in Formula (20),

$$T=(N-1)^{p-1}(N-1-p)N^{N-2-p} \tag{22}$$

is obtained. Formula (22) gives the total number of trees in incomplete graphs belonging to *p-Series*[2], one of the basic series in incomplete graphs derived from a complete graph.

Furthermore, for $k_1=k_2=---=k_p=n$ and $p=N\text{-}2$, Formula (20) becomes

$$T=n^{N-1}(N-1)^{N-3}=n[n\{n(N-1)\}^{N-3}]. \tag{23}$$

The equation shown within the brackets [ ] of Formula (23) expresses the total number of trees in a full graph of N-1 nodes[1]. From this fact, we can obtain easily Formula (23) by observing the graphs which is satisfied the conditions stated above.

4.3 Specific series derived from *generalized m-Series*

The expressions for D_m obtained from Eq. (11) by putting $k_1=k_2=-----=k_m\equiv k$ is shown in Table 1.

Table 1 Expressions for D_m

m	D_m
3	$nN\cdot(nN-3k)^2$
4	$nN\cdot(nN-2k)^2(nN-4k)$
5	$nN\cdot\{(nN)^2-5k(nN)+5k^2\}^2$
6	$nN\cdot\{(nN)^2-4k(nN)+3k^2\}^2(nN-4k)$
7	$nN\cdot\{(nN)^3-7k(nN)^2+14k^2(nN)-7k^3\}^2$
8	$nN\cdot\{(nN)^3-6k(nN)^2+10k^2(nN)-4k^3\}^2(nN-4k)$
9	$nN\cdot\{(nN)^4-9k(nN)^3+27k^2(nN)^2-30k^3(nN)+9k^4\}^2$
10	$nN\cdot\{(nN)^4-8k(nN)^3+21k^2(nN)^2-20k^3(nN)+5k^4\}^2(nN-4k)$
·	·
·	·
·	·

In order to give an unificative expression of D_m in Table 1, we now define the following function:

$$F(x)=\sum_{\gamma=1}^{x/2}[\{K(x)/(\gamma-1)!\}(-k)^{\gamma-1}\cdot(nN)^{x/2-\gamma}, \tag{24}$$

where x is even and

$$K(x)=\begin{cases}1 & \text{for } \gamma=1,\\ \prod_{\lambda=\gamma}^{2(\gamma-1)}(x-\lambda) & \text{for } \gamma\geqq 2.\end{cases} \tag{25}$$

From Eqs.(24) and (25), D_m is expressed by the following equations:

$$D_m=\begin{cases}nN\{F(m+1)-k\cdot F(m-1)\}^2, & (m \text{ odd}),\\ nN(nN-4k)\{F(m)\}^2, & (m \text{ even}).\end{cases} \tag{26}$$

Consequently, from Formula (13), the total number of trees in the graphs belonging to this specific series, which is satisfied the condition of $k_1=k_2=---=k_m\equiv k$, is given by

$$T=\begin{cases}n\{F(m+1)-k\cdot F(m-1)\}^2\cdot(nN)^{N-1-m}, & (m \text{ odd}),\\ n(nN-4k)\{F(m)\}^2\cdot(nN)^{N-1-m}, & (m \text{ even}).\end{cases} \tag{27}$$

Let $k=n$. In this case, Eq.(24) becomes

$$F(x)\,|_{k=n}=n^{\frac{x}{2}-1}\cdot\sum_{\gamma=1}^{x/2}[(-1)^{\gamma-1}\{K(x)/(\gamma-1)!\}N^{\frac{x}{2}-\gamma}]=n^{\frac{x}{2}-1}\cdot F_0(x)\,. \quad (28)$$

We thus obtain the total number of trees for $k=n$ as

$$T=\begin{cases} n^{N-1}\{F_0(m+1)-F_0(m-1)\}^2\cdot N^{N-1-m}\,, & (m \text{ odd}),\\ n^{N-1}(N-4)\{F_0(m)\}^2\cdot N^{N-1-m}\,, & (m \text{ even}).\end{cases} \quad (29)$$

The total number of trees for $k_1=k_2=\text{----}=k_m=n=1$ is obtained from Formula (29) by putting n=1. It gives the general formula for the total number of trees in the graphs belonging to *m-Series*, one of the basic series in incomplete graphs derived from a complete graph, and $\{F_0(m+1)-F_0(m-1)\}^2$ and $(N-4)\{F_0(m)\}^2$ in Formula (29) also give the unified expressions of generic factor[1] in that series.

4.4 Specific series derived from *generalized s-Series*

We now represent the equation expressing within the brackets [ ] of Eq.(14) by a symbol D_s. Table 2 shows the expressions for D_s obtained by putting $k_1=k_2=\text{----}=k_s\equiv k$.

Table 2 Expressions for D_s

s	D_s
1	$nN\cdot(nN-2k)$
2	$nN\cdot(nN-3k)(nN-k)$
3	$nN\cdot\{(nN)^2-4k(nN)+2k^2\}(nN-2k)$
4	$nN\cdot\{(nN)^2-5k(nN)+5k^2\}\{(nN)^2-3k(nN)+k^2\}$
5	$nN\cdot\{(nN)^2-4k(nN)+k^2\}\{(nN)^2-4k(nN)+3k^2\}(nN-2k)$
6	$nN\cdot\{(nN)^3-7k(nN)^2+14k^2(nN)-7k^3\}\{(nN)^3-5k(nN)^2+6k^2(nN)-k^3\}$
7	$nN\cdot\{(nN)^2-4k(nN)+2k^2\}\{(nN)^4-8k(nN)^3+20k^2(nN)^2-16k^3(nN)+2k^4\}$ $\cdot(nN-2k)$
8	$nN\cdot\{(nN)^4-9k(nN)^3+27k^2(nN)^2-30k^3(nN)+9k^4\}\{(nN)^4-7k(nN)^3$ $+15k^2(nN)^2-10k^3(nN)+k^4\}$
9	$nN\cdot\{(nN)^4-8k(nN)^3+19k^2(nN)^2-12k^3(nN)+k^4\}\{(nN)^4-8k(nN)^3$ $+21k^2(nN)^2-20k^3(nN)+5k^4\}(nN-2k)$
10	$nN\cdot\{(nN)^5-11k(nN)^4+44k^2(nN)^3-77k^3(nN)^2+55k^4(nN)-11k^5\}$ $\cdot\{(nN)^5-9k(nN)^4+28k^2(nN)^3-35k^3(nN)^2+15k^4(nN)-k^5\}$
·	·
·	·
·	·

The unificative expression of D_s in Table 2 is also given by using the function defined in Eq.(24), that is

$$D_s=\begin{cases} nN\cdot F(s+1)\{F(s+3)-k^2\cdot F(s-1)\}\,, & (s \text{ odd}),\\ nN\cdot\{F(s+2)+k\cdot F(s)\}\{F(s+2)-k\cdot F(s)\}, & (s \text{ even}).\end{cases} \quad (30)$$

Consequently, from Formula (14), the total number of trees in the graphs belonging to this specific series, which is satisfied the condition of $k_1=k_2=\cdots=k_s\equiv k$, is given by

$$T=\begin{cases} n\cdot F(s+1)\{F(s+3)-k^2\cdot F(s-1)\}(nN)^{N-2-s}, & (s \text{ odd}), \\ n\cdot\{F(s+2)+k\cdot F(s)\}\{F(s+2)-k\cdot F(s)\}(nN)^{N-2-s}, & (s \text{ even}). \end{cases} \quad (31)$$

If it is assumed that $F(0)=0$, Eq.(31) can be also applied for $s=1$.

Let $k=n$. In this case, from Eqs.(28) and (31), we have

$$T=\begin{cases} n^{N-1}\cdot F_0(s+1)\{F_0(s+3)-F_0(s-1)\}\cdot N^{N-2-s}, & (s \text{ odd}), \\ n^{N-1}\cdot\{F_0(s+2)+F_0(s)\}\{F_0(s+2)-F_0(s)\}\cdot N^{N-2-s}, & (s \text{ even}), \end{cases} \quad (32)$$

where it is assumed that $F_0(0)=0$.

The general formula for the total number of trees in the graphs belonging to *s-Series*, one of the basic series in incomplete graphs derived from a complete graph, is obtained from Formula (32) by putting $n=1$. In this case, $F_0(s+1)\{F_0(s+3)-F_0(s-1)\}$ and $\{F_0(s+2)+F_0(s)\}\{F_0(s+2)-F_0(s)\}$ give the unified expressions of generic factor[1] in that series.

5. Examples

In the present chapter, some examples of the graphs belonging to four generalized series discussed in this paper are shown. These examples are arranged in Figs. 6, 7, 8 and 9.

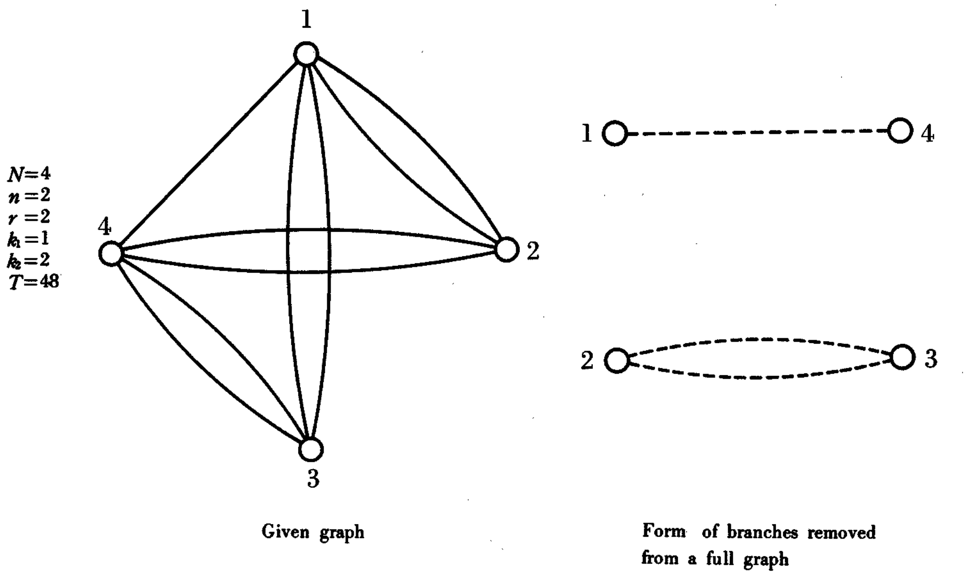
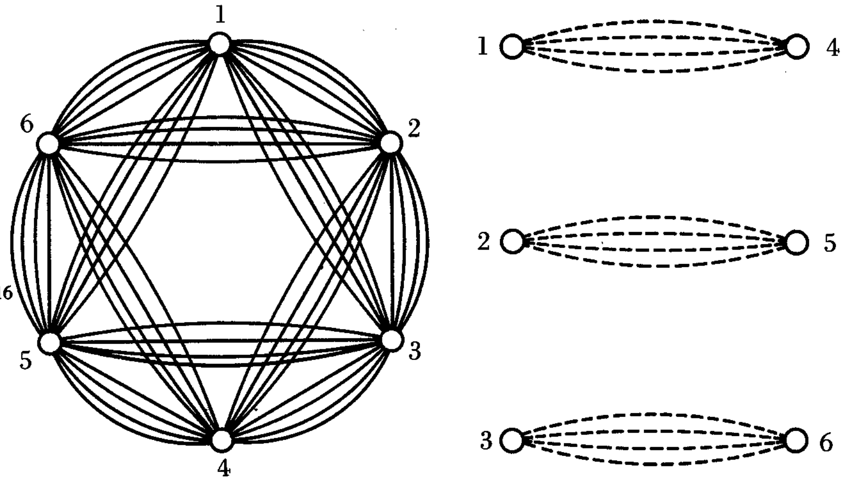


Fig. 6 (a) Example of the graphs belonging to *generalized r-Series.*

$N=6$
$n=4$
$r=3$
$k_1=k_2$
$=k_3=4$
$T=393,216$

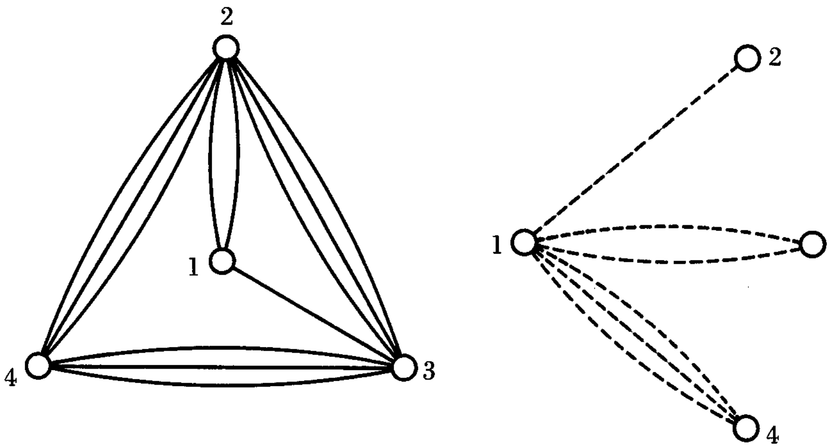


Given graph

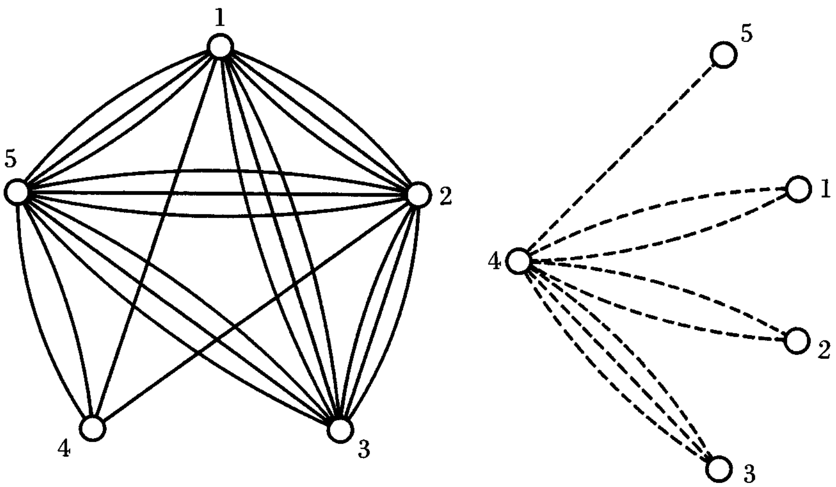
Form of branches removed from a full graph

Fig. 6 (b) Example of the graphs belonging to *generalized r−Series*.

(a) $N=4$
$n=3$
$p=3$
$k_1=1$
$k_2=2$
$k_3=3$
$T=93$



(b) $N=5$
$n=3$
$p=4$
$k_1=1$
$k_2=k_3=2$
$k_4=3$
$T=2,106$



Given graphs

Forms of branches removed from a full graph

Fig. 7 Examples of the graphs belonging to *generalized p-Series*.

(a) $N=4$
$n=3$
$m=4$
$k_1=1$
$k_2=k_3=2$
$k_4=3$
$T=59$

(b) $N=6$
$n=2$
$m=5$
$k_1=k_3=1$
$k_2=k_4$
$=k_5=2$
$T=7,192$

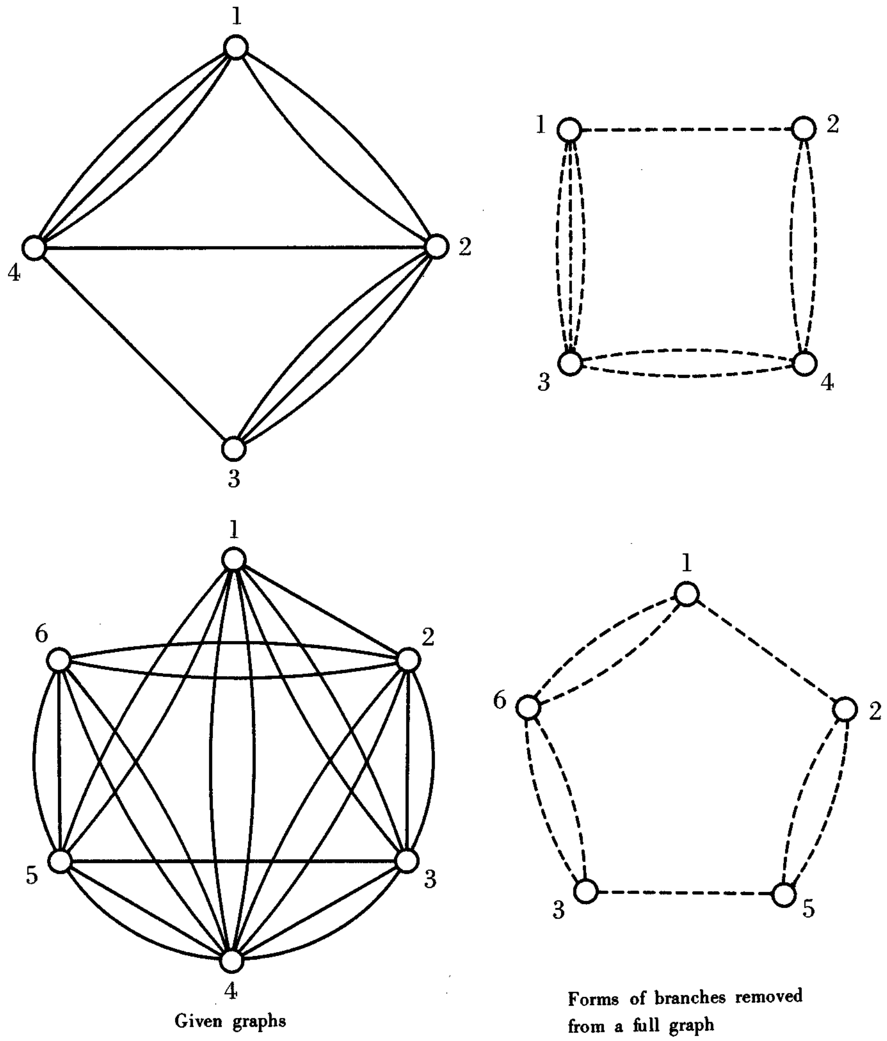


Fig. 8 Examples of the graphs belonging to *generalized m-Series.*

$N=4$
$n=3$
$s=3$
$k_1=2$
$k_2=1$
$k_3=3$
$T=111$

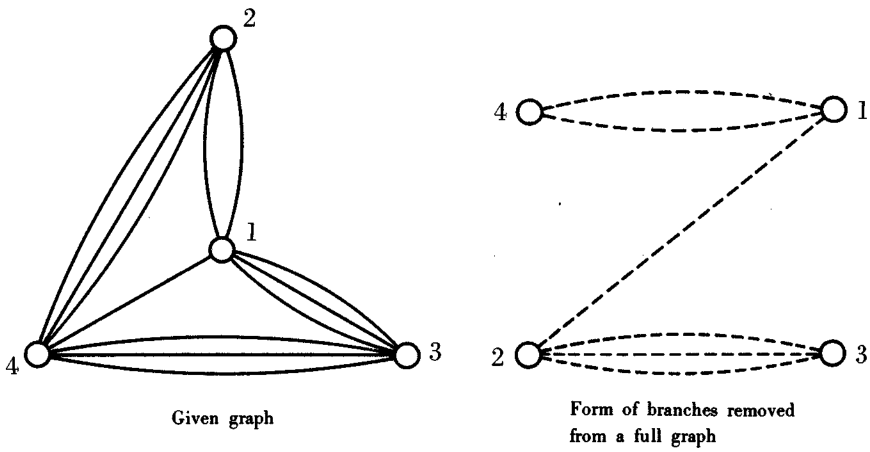


Fig. 9 (a) Example of the graphs belonging to *generalized s-Series.*

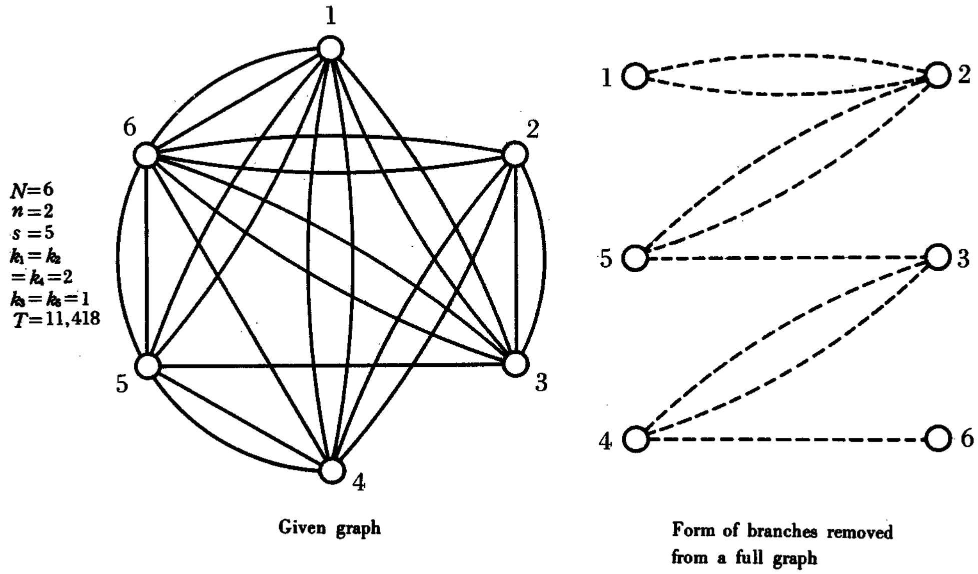


Fig. 9 (b) Example of the graphs belonging to *generalized s–Series.*

6. Conclusions

In the above, we have investigated the total number of trees in four kinds of the graphs derived from a full graph of N nodes by removing four different sets of branches.

The formulas for the total number of trees in the graphs belonging to the four series have been obtained and some specific series derived from these series, as special cases, have been discussed.

We can describe a few interesting results, the most basic of which are as follows:

(1) Two series named *m*-and *s-Series* in the basic series of incomplete graphs are treated as independent series. The *generalized m–* and *generalized s-Series* in the present paper, however, are not independent, that is, the latter is derived from the former by giving the special conditions.

(2) The total number of trees in the graphs belonging to *generalized m–* and *generalized s-Series* can be obtained easily by making the specific diagrams representing the conditions which are regulated by the number of branches to be removed from an original full graph in their graphs.

(3) The four basic series which are called *r-*, *p-*, *m-* and *s-Series* in incomplete graphs are derived from each of the four generalized series which were investigated in this paper.

References

1) S.D. Bedrosian, J. Franklin Inst., **277**, 313 (1964).
2) L. Weinberg, Proc. IRE, **46**, 1954 (1958).
3) S.D. Bedrosian, IRE Trans., **CT-8**, 363 (1961).
4) T. Kasai et al, J.I.E.C.E., Japan, **49**, 69 (1966).