Natural Convection between Horizontal Concentric Cylinders for Low Grashof Numbers

Natural Convection between Horizontal Concentric Cylinders for Low Grashof Numbers*

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In the present paper is obtained a solution for two-dimensional steady natural convection of a viscous fluid between two horizontal concentric cylinders maintained at different uniform temperatures. The analysis is based on the Boussinesq approximation for buoyancy, the effects of viscous and compressional heatings being neglected. The solutions are expanded in power series of Grashof number and the governing equations are integrated by a method of successive approximation. The complex variable $z=x+iy$ and its complex conjugate $\bar{z}=x-iy$ are introduced instead of the usual rectangular coordinates variables x and y, which makes integration of the governing equations and determination of complementary functions systematic and easy. The streamlines are calculated taking first two terms in the series solution for stream function and they are qualitatively in good agreement with those photographed by Bishop and Carley in their experiments. The isotherms are also calculated in the similar procedure. Both the local and the overall heat transfer rates are obtained in the form of Nusselt numbers and are compared with those in the case of pure conduction.

1. Introduction

Although natural convection within an enclosed space is an interesting and important problem, not so many investigations have been done so far. However, for the particular case of the natural convection between two horizontal concentric cylinders kept at different uniform temperatures, photographic and qualitative descriptions of the convective flow were recently presented by Bishop and Carley[1]), and numerical solutions of the governing equations were given by Crawford and Lemlich[2]), as well as an analytical solution to the Stokes approximation. More recently, Mack and Bishop[3]) obtained an analytical solution by expanding the temperature and the stream functions in power series of Rayleigh number.

The present analysis is undertaken to give a theoretical interpretation to the photographs of flow pattern by Bishop and Carley, independently of Mack and Bishop's analysis. It starts with expanding the stream function and the temperature distribution in power series of Grashof number like Mack and Bishop's analysis did. In the present analysis, however, the governing equations are expressed in the complex variables z and $\bar{z}$, which makes integration of the governing equations and determination of the complementary functions more systematic and easier than in their analysis.

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In section 3 are discussed the particular features of the solution which can be seen in the configuration of streamlines, the temperature distribution and the distribution of the local heat transfer rates on the surface of the cylinders. The effects of Grashof and Prandtl numbers and of radius ratio of the two cylinders upon these are also discussed. The calculated streamline configurations are compared with those photographed by Bishop and Carley.

2. Mathematical formulation of the problem and method of solution

2.1 Governing equations

We consider two-dimensional steady natural convection of a viscous fluid between two horizontal concentric cylinders which are kept at different uniform temperatures, say, the inner cylinder being kept hotter. The coordinates system is taken as shown in Fig. 1. The flow and the temperature distributions are supposed to be symmetrical with respect to the vertical plane through the axis of the cylinders.

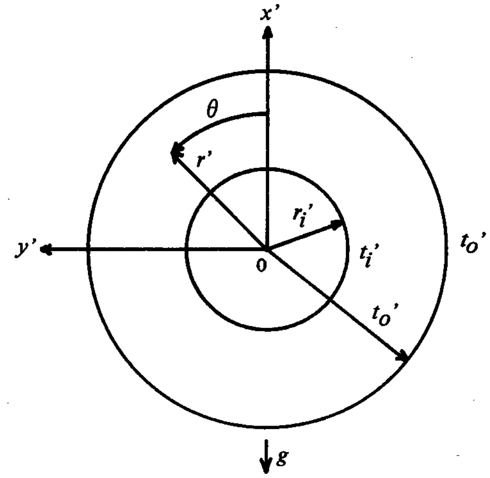

Fig. 1 Coordinates for this problem.

The temperature difference of the two cylinders is assumed to be so small that the terms of work by compression and of viscous dissipation can be neglected in the energy equation. The Boussinesq approximation will also hold good; all physical properties of the fluid are considered to be independent of temperature except density in buoyancy term in Navier-Stokes equations and variation of the density is approximated by a linear function of variation of temperature. Thus, the system of the governing equations for the present problem consists of equation of continuity, Navier-Stokes equations and energy equation which are quite the same in their forms as the usual ones for an incompressible flow except the additional term of buoyancy in Navier-Stokes equations.

Now, we shall express the governing equation in non-dimensional forms.

The non-dimensional temperature T is defined as

$$T = (t' - t'_o)/(t'_i - t'_o),$$

where t' is the dimensional temperature, t'_i and t'_o being the temperatures of the inner and the outer cylinders, respectively. All other dimensional quantities are made non-dimensional as usual by choosing appropriate reference length and velocity. For the former we can choose the radius r'_i of the inner cylinder, but for the latter no representative velocity is given in the present problem of natural convection. Combining the parameters which take part in natural convection under consideration, we can build up a reference velocity, as

$$U = g\beta(t'_i - t'_o)\, r'^2_i/\nu,$$

where g is the acceleration due to gravity, β and ν being the coefficient of thermal expansion and the kinematic viscosity of the fluid, respectively. Introducing the non-dimensional stream function from which the non-dimensional velocity components u and v are derived as

$$u = -\partial\Psi/\partial y \quad \text{and} \quad v = \partial\Psi/\partial x,$$

the governing equations are expressed in the following non-dimensional forms;

$$Gr\left(-\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)\Delta\Psi = \Delta\Delta\Psi - \frac{\partial T}{\partial y}, \tag{1}$$

and

$$GrPr\left(-\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x} + \frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}\right)T = \Delta T, \tag{2}$$

where the operator Δ stands for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, x and y are the non-dimensional coordinates variables, Gr and Pr being the Grashof and the Prandtl numbers, respectively, defined as

$$Gr = g\beta(t'_i - t'_o)\, r'^3_i/\nu^2 \quad \text{and} \quad Pr = \nu/\alpha,$$

where α is the thermal diffusivity of the fluid. The boundary conditions for the present problem are

$$-\partial\Psi/\partial y = \partial\Psi/\partial x = 0 \quad \text{at} \quad r = 1, \tag{3}$$

$$-\partial\Psi/\partial y = \partial\Psi/\partial x = 0 \quad \text{at} \quad r = R, \tag{4}$$

$$T = 1 \quad \text{at} \quad r = 1, \tag{5}$$

and

$$T = 0 \quad \text{at} \quad r = R, \tag{6}$$

where R is the non-dimensional radius of the outer cylinder or the radius ratio of the two cylinders.

Equation (1) is the vorticity equation obtained by eliminating the pressure terms from Navier-Stokes equations with the additional buoyancy term. The equation of continuity is automatically satisfied by using the stream function. Equation (2) is energy equation in which the terms of viscous dissipation and of work of compression are neglected. Conditions (3) and (4) are no-slip conditions on the solid boundaries and conditions (5) and (6) show that the inner and the outer cylinders are kept at given uniform temperatures.

2.2 Method of solution

The governing equations (1) and (2) constitute a set of non-linear equations coupled with one another, which makes the exact solution of the present problem impossible. However, when the temperature difference is small and so the induced convective flow is slow, we can assume that the Grashof number is so small that Ψ and T can be expanded in power series of the Grashof number, as

$$\Psi = \sum_{j=o}^{\infty} Gr^j \Psi_j \tag{7}$$

and

$$T = \sum_{j=o}^{\infty} Gr^j T_j. \tag{8}$$

Substituting (7) and (8) into the governing equations (1) and (2) and equating the coefficients of the same power of Gr, we have an infinite set of uncoupled linear equations. Introducing the complex variable $z=x+iy$ and its complex conjugate $\bar{z}=x-iy$, the differential operators Δ and $\Delta\Delta$ in equations (1) and (2) are written in the forms,

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}} ,$$

$$\Delta\Delta \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 = 16\frac{\partial^4}{\partial z^2 \partial \bar{z}^2} ,$$

Then, the above obtained infinite set of linear equations can be written down in terms of z and $\bar{z}$;

$$4\frac{\partial^2 T_o}{\partial z \partial \bar{z}} = 0, \tag{9}$$

$$16\frac{\partial^4 \Psi_o}{\partial z^2 \partial \bar{z}^2} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) T_o , \tag{10}$$

$$4\frac{\partial^2 T_1}{\partial z \partial \bar{z}} = 2iPr\left(\frac{\partial \Psi_o}{\partial \bar{z}}\frac{\partial}{\partial z} - \frac{\partial \Psi_o}{\partial z}\frac{\partial}{\partial \bar{z}}\right) T_o, \tag{11}$$

$$16\frac{\partial^4\Psi_1}{\partial z^2\partial\bar{z}^2}=i\left\{8\left(\frac{\partial\Psi_O}{\partial z}\frac{\partial}{\partial z}-\frac{\partial\Psi_O}{\partial z}\frac{\partial}{\partial\bar{z}}\right)\frac{\partial^2\Psi_O}{\partial z\partial\bar{z}}+\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial\bar{z}}\right)T_1\right\}, \tag{12}$$

- ,

subject to the boundary conditions,

$$T_o = 1 \qquad \text{at} \quad z\bar{z} = 1, \tag{13}$$

$$T_o = 0 \qquad \text{at} \quad z\bar{z} = R^2, \tag{14}$$

$$T_j = 0 \qquad \text{at} \quad z\bar{z} = 1, R^2, j \geqq 1, \tag{15}$$

and

$$\frac{\partial\Psi_j}{\partial z}=\frac{i}{2}(u_j-iv_j)=0 \qquad \text{at} \quad z\bar{z} = 1, R^2, j \geqq 0, \tag{16}$$

where u_j and v_j are the x- and the y-components of the perturbed velocity of the jth order, respectively.

Now, we have only to solve these linear equations successively. We shall present the method of solution for equation (10) in detail, as an example, and only the results for the others.

The solution of equation (9) subject to conditions (13) and (14) is given by

$$T_o = 1 - \frac{\log z\bar{z}}{2\log R}. \tag{17}$$

This solution is the well-known temperature distribution for pure conduction, that is, the temperature distribution which would occur if the fluid were immobile. Substituting (17) into the right hand side of (10) and carrying out integration with respect to z and $\bar{z}$, we obtain

$$\Psi_o=\frac{i}{32\log R}\left\{\frac{1}{2}z^2\bar{z}(\log\bar{z}-1)-\frac{1}{2}z\bar{z}^2(\log z-1)\right\}$$
$$+\bar{z}F_1(z)+zF_2(\bar{z})+F_3(z)+F_4(\bar{z}), \tag{18}$$

where F_1, F_2, F_3 and F_4 are complementary functions to be determined by the boundary conditions (16).

These complementary functions must be subject to the following conditions in addition to the boundary conditions (16).

Condition (i): Ψ_O must be real, so that the following relations must be satisfied,

$$F_2(\bar{z})=\bar{F}_1(\bar{z}) \quad \text{and} \quad F_4(\bar{z})=\bar{F}_3(\bar{z}). \tag{19}$$

Condition (ii): the flow field is symmetrical with the x-axis, so that the stream function should be subject to the relations,

$$\Psi(z,\bar{z}) = -\Psi(\bar{z},z). \tag{20}$$

Condition (iii): $\dfrac{\partial\Psi}{\partial z}$ and Ψ must be single-valued functions, so that the complementary functions must contain some multi-valued logarithmic functions to cancel the multi-valuedness of the principal part of (18).

Transforming $F_1(z)$ and $F_3(z)$ into $F(z)$ and $G(z)$ by the relations,

$$F_1(z) = \frac{i}{32\log R}F(z) \quad \text{and} \quad F_3(z) = \frac{i}{32\log R}G(z), \tag{21}$$

and applying the condition (i) to $F_2(\bar{z})$ and $F_4(\bar{z})$, we have

$$\Psi_o = \frac{i}{32\log R}\left\{\frac{1}{2}z^2\bar{z}(\log\bar{z}-1) - \frac{1}{2}z\bar{z}^2(\log z-1)\right. \\ \left. + \bar{z}F(z) - zF(\bar{z}) + G(z) - G(\bar{z})\right\}. \tag{22}$$

Next, the conditions (ii) and (iii) being applied to (22), we obtain

$$\Psi_o = \frac{i}{32\log R}\left\{\frac{1}{2}z^2\bar{z}(\log z\bar{z}-2) - \frac{1}{2}z\bar{z}(\log z\bar{z}-2)\right. \\ \left. + \bar{z}F(z) - zF(\bar{z}) + G(z) - G(\bar{z})\right\}, \tag{23}$$

where the multi-valuedness of the pricipal part of (22) is eliminated by adding the appropriate complementary functions.

Differentiating the equation (23) with respect to z, we have

$$\frac{\partial\Psi_o}{\partial z} = \frac{i}{32\log R}\left\{z\bar{z}(\log z\bar{z}-2) + \frac{1}{2}z\bar{z} - \frac{1}{2}\bar{z}^2(\log z\bar{z}-2)\right. \\ \left. -\frac{1}{2}\bar{z}^2 + \bar{z}\frac{dF(z)}{dz} - F(\bar{z}) + \frac{dG(z)}{dz}\right\}. \tag{24}$$

The boundary conditions (16) are imposed on (24) and the complementary functions are finally determined as

$$\bar{z}dF(z)/dz = Az\bar{z} + B\bar{z}/z\ , \tag{25}$$

$$-F(\bar{z}) = -A\bar{z}^2/2 - B\log\bar{z}\ , \tag{26}$$

and
$$dG(z)/dz = -B\log z + C + D/z^2\ , \tag{27}$$

where A, B, C and D are the real functions of the radius ratio R. The solutions for the other Ψ_j's and T_j's are to be obtained by the similar methods.

2.3 Solutions

For the sake of convenience of numerical computation, the solutions obtained above are expressed in the polar coordinates variables (r, θ):

$$T_o = 1 - \frac{\log r}{\log R}, \tag{28}$$

and

$$\Psi_o = -\frac{r\sin\theta}{16\log R}\left[\{r^2 - f_1(R)\}\log r + f_2(R)r^2 + f_3(R) + f_4(R)\frac{1}{r^2}\right], \tag{29}$$

where

$$f_1(R) = \frac{(1-R^2)\{(1-R^4)+4R^2\log R\}}{2\{(1-R^2)^2+(1-R^4)\log R\}}, \tag{30}$$

$$f_2(R) = -\frac{(1-R^2)^2+2(1-R^2)^2\log R - 4R^4(\log R)^2}{4\{(1-R^2)^2+(1-R^4)\log R\}} \tag{31}$$

$$f_3(R) = \frac{(1-R^2)(1-R^4)+2(1-R^2)^2\log R - 8R^4(\log R)^2}{4\{(1-R^2)^2+(1-R^4)\log R\}}, \tag{32}$$

and

$$f_4(R) = \frac{-R^2(1-R^2)^2+4R^4(\log R)^2}{4\{(1-R^2)^2+(1-R^4)\log R\}}. \tag{33}$$

These solutions are in perfect agreement with the creeping-flow solution obtained by Crawford and Lemlich.

The solutions to the second approximation are

$$T_1 = Pr\frac{2r\cos\theta}{(16\log R)^2}\left[f_5(R)(\log r)^2 - \left\{r^2 + f_6(R) + f_7(R)\frac{1}{r^2}\right\}\log r + f_8(R)r^2 + f_9(R) + f_{10}(R)\frac{1}{r^2}\right] \tag{34}$$

and

$$\Psi_1 = \frac{r^2\sin 2\theta}{(32\log R)^2}\left[r^4\left\{\frac{1}{12}(\log r)^2 + f_{11}(R,Pr)\log r + f_{12}(R,Pr)\right\} + r^2\{f_{13}(R,Pr)(\log r)^2 + f_{14}(R,Pr)\log r + f_{15}(R,Pr)\} + f_{16}(R,Pr)(\log r)^2 + f_{17}(R,Pr)\log r + f_{18}(R,Pr) + f_{19}(R,Pr)\frac{1}{r^2} + f_{20}(R,Pr)\frac{1}{r^4}\right], \tag{35}$$

where $f_m(R)$ and $f_n(R, Pr)$ are the real functions of R and of R and Pr, respectively. The detailed expressions for the coefficients in (34) and (35) are too lengthy to be reproduced here and are omitted to conserve space.

3. Discussion

Although we considered the inner cylinder to be hotter in the formulation of the problem, the solution in the case in which the outer cylinder is hotter can be obtained from our solution merely by use of negative values of Grashof number.

In the present analysis it is convenient to use a Grashof number based on the radius of the inner cylinder, but a Grashof number Gr^* based on the distance r_o'-r_i' between the cylinders is commonly used in experimental work and it is related to our Gr by $Gr^*=(R^3-1)\,Gr$.

3.1 Range of Grashof number for which solution is reliable

We postulate that Ψ and T can be expanded in asymptotic power series of Gr for small values of Gr. However, it seems from the results of numerical computations that the solution is sufficiently reliable even for $Gr=1000$ in the case of $R=2$ and $Pr=0.7$, where Ψ_1/Ψ_o and T_1/T_o are of order 10^{-4}. The upper limit of Grashof number within the reliability of solution decreases with increase of the radius ratio and changes little with Prandtl number.

3.2 Streamlines and isotherms configurations

The streamlines and isotherms configurations are calculated taking first two terms in Ψ and T, and are shown in Fig. 2 for $Gr=1000, Pr=0.7$ (approximate value for air) and $R=2$.

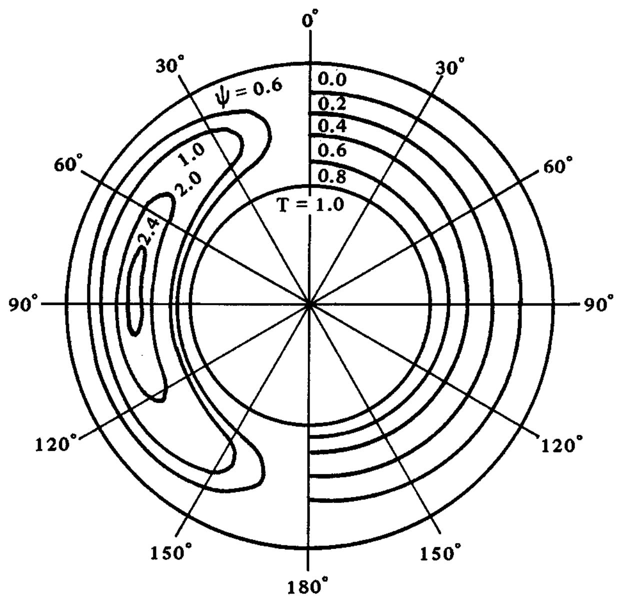

Fig. 2 Streamlines and isotherms configurations for Gr = 1000, Pr = 0.7 and R = 2.00

The creeping-flow stream function Ψ_0 has its maximum value at $\theta=90°$, and the streamlines configuration in the upper half of the flow region is a mirror image of that in the lower half; this configuration is independent of Prandtl number. However, the solution to the second approximation presents that, as the Grashof number increases, the centre of the eddy rises (from $\theta=90°$) into the upper half of the flow region. A change in the Prandtl number has very little qualitative effect upon the streamlines.

The isotherms configuration in the case of pure conduction is a set of concentric circles. On the other hand, the solution to the second approximation presents that the isotherms are not concentric; each of the isothermal lines is deformed from a right circle and shift upper. Correspondingly, the isotherms are thicker near the lower side of the inner cylinder and the upper side of the outer cylinder, and thinner near the upper side of the inner cylinder and the lower side of the outer cylinder. This tendency becomes more considerable as the Grashof number and the Prandtl number increase.

3.3 Heat transfer rates

We express the local radial heat-flow rates per unit area at the inner and the outer cylinders by means of the corresponding local Nusselt numbers $Nu_i(\theta)$ and $Nu_o(\theta)$,

$$Nu_i = -\left(r\frac{\partial T}{\partial r}\right)_{r=1} \tag{36}$$

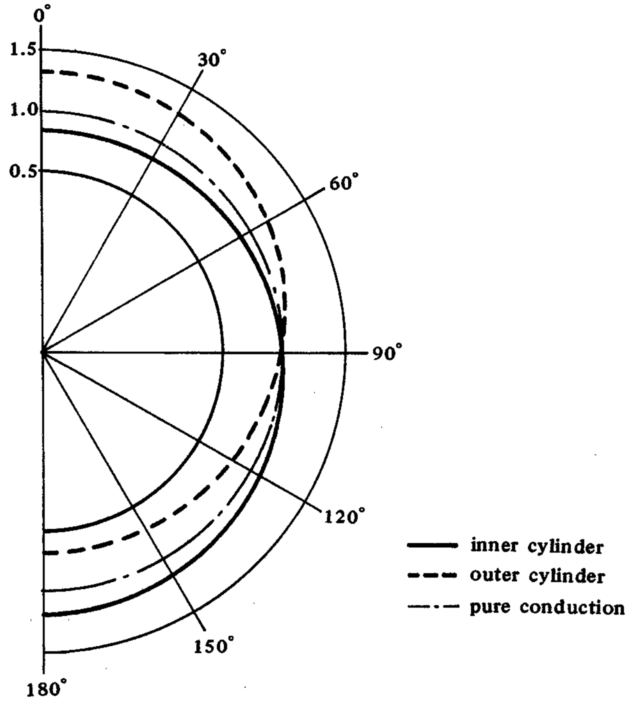

Fig. 3 $Nu/Nu^{(0)}$–distributions for Gr = 1000, Pr = 0.7 and R = 2.00

and
$$Nu_o = -\left(r\frac{\partial T}{\partial r}\right)_{r=R} . \tag{37}$$

Similarly, we express the overall heat-flow rate from the inner clinder to the outer cylinder by means of the overall Nusselt number $\overline{Nu}$,

$$\overline{Nu} = -\frac{1}{\pi}\int_o^{\pi}\left(r\frac{\partial T}{\partial r}\right)_{r=1} d\theta = -\frac{1}{\pi}\int_o^{\pi}\left(r\frac{\partial T}{\partial r}\right)_{r=R} d\theta . \tag{38}$$

Fig. 3 shows the $Nu_i/Nu_i^{(0)}$- and $Nu_o/Nu_o^{(0)}$-distributions, $Nu_i^{(0)}$ and $Nu_o^{(0)}$ being the Nusselt number for the pure conduction at the inner and the outer cylinders, respectively. For the inner cylinder, the Nusselt number takes larger value at the lower side and smaller value at the upper side than that in the case of pure conduction. The reverse is the case for the outer cylinder. This is evident from the isotherms configuration. This tendency becomes more considerable as the Grashof number and the Prandtl number increase.

Examination of the perturbation equations proves that the influences of Grashof number and of Prandtl number on the overall Nusselt number are of higher order and come out of the terms of the third and higher powers of Grashof number.

4. Conclusion

The solution of the governing equations for steady convection of a viscous fluid between the two horizontal concentric cylinders kept at the different temperatures has been obtained as far as the first two terms both in the series expansion of temperature and in that of stream function in powers of Grashof number. The new method of solution is presented, in which the complex coordinates variables z and $\bar{z}$ are introduced and the analysis of complex functions is applied. It makes integration of the linearized governing equations and determination of the complementary functions systematic and easy.

For the value of Prandtl number corresponding to air, the streamlines configurations are of the 'crescent-eddy' type in accordance with the experimental evidence given by Bishop and Carley.

Both the local and the overall heat-flow rates are expressed by means of the corresponding Nusselt numbers. The variation of the local Nusselt numbers with angular position is compared with that for the pure conduction. The influences of Grashof number and Prandtl number on the overall heat transfer is a higher-order effect coming out of the third and the higher approximations.

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