A Note on the Longest Path Problem

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The loop-free longest path problem in graph theory is of practical importance in operations research problems. In this paper an algebraic system fit for the analysis of this problem is proposed and the analysis is carried out with the aid of this algebra.

The application of the algebra to this problem leads us to various results of significance, such as, (1) it is able to clarify that, from the mathematical point of view, the loop-free longest path problem may be regarded as a class of communication system problems, (2) diakoptical formulae for this problem can be derived, which are extremely effective for the analysis of large scale systems.

1. Introduction

The loop-free longest paths in graphs play an important role in operations research problems related to scheduling, critical paths and project networks. But, as far as authors know, the research on the method of determination of the loop-free longest path can not be found except that carried out by Y.C.Chen and O.Wing. Perhaps, such a state must be resulted from the fact that, on the graph-theoretic basis, the analysis of the loop-free longest paths could be executed in the similar way that of the shortest paths on which several methods were proposed by many researchers[2-9]. Formerly, one of the authers studied an algebraic theory of network-theoretical problems which arise in communication systems, that is, problem of reliability, maximum capacity, the shortest paths, the most reliable paths, the maximum capacity paths, and of connectivity[9]. Comparing with the graph-theoretic method, the analysis on algebraic bases has several merits. First, it is suitable for the use of a digital computer because a digital computer does not always fit for memorizing and treating graphs. Second, it enables us to express all problems by the same-type formulae. Third, diakoptical formulae for large scale systems can be derived in refined forms, and these serve for reducing computation time. In the present situation of the research on the loop-free longest paths, it should never be worthless to point out the possibility of an algebraic treatment of this problem.

2. The Longest Path Problem and Algebraic System

Let e_i be an element of a graph, a_i be the physical quantity (distance) imposed on e_i and $\mathbf{a}_i=(e_i\ , a_i)$ be a composite quantity. An algebra as the

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mathematics for expressing and analysing the longest path problem can be defined by the following axioms;

(1) Operations γ and λ satisfying the following postulates are defined between elements of $\{a_i\}$.

1. $a_i \gamma a_j = a_j \gamma a_i$, $\quad a_i \lambda a_j = a_j \Lambda a_i$,
2. $a_i \gamma (a_j \gamma a_k) = (a_i \gamma a_j) \gamma a_k$, $\quad a_i \lambda (a_j \lambda a_k) = (a_i \lambda a_j) \lambda a_k$,
3. $a_i \gamma (a_i \lambda a_j) = a_i$, $\quad a_i \lambda (a_i \gamma a_j) = a_i$,
4. $a_i \gamma (a_j \lambda a_k) = (a_i \gamma a_j) \lambda (a_i \gamma a_k)$, $\quad a_i \lambda (a_j \gamma a_k) = (a_i \lambda a_j) \gamma (a_i \lambda a_k)$,
5. $a_i \gamma a_i = a_i$, $\quad a_i \lambda a_i = a_i$,
6. $\exists a_o, a^\circ$, for $\forall a_i$ such that
$$a_i \gamma a_o = a_i,\ a_i \gamma a^\circ = a^\circ,\ a_i \lambda a_o = a_i,\ a_i \lambda a^\circ = a^\circ$$

(2) Operations V and Λ satisfying the following postulates are defined between elements of $\{a_i\}$.

1. $a_i \vee a_j = a_j \vee a_i$, $\quad a_i \wedge a_j = a_j \wedge a_i$,
2. $a_i \vee (a_j \vee a_k) = (a_i \vee a_j) \vee a_k$, $\quad a_i \wedge (a_j \wedge a_k) = (a_i \wedge a_j) \wedge a_k$,
3. $a_i \vee (a_j \wedge a_k) = (a_i \vee a_j) \wedge (a_i \vee a_k)$, $\quad a_i \wedge (a_j \vee a_k) = (a_i \wedge a_j) \vee (a_i \wedge a_k)$,
4. $\quad a_i \wedge a_i = a_i$,
5. $\quad a_i \wedge (a_i \vee a_j) = a_i \vee a_j$,
6. $\exists a^\circ, a_o$ for $\forall a_i$ such that
$$a_i \vee a_o = a_i,\ a_i \vee a^\circ = a^\circ,\ a_i \wedge a_o = a_i,\ a_i \wedge a^\circ = a^\circ$$

(3) There exist following relations between a°, a_o, e°, e_o, $\bar{a}^\circ$, and a_o.

1. $(e_i, a_i = a^\circ) = a^\circ \quad (e_i, a_i = a_o) = a_o$
2. $(e_i = e^\circ, a_i) = a^\circ$, $(e_i = e_o, a_i) = a_o$

Axioms (1) are these of a distributive lattice. Let A and B be nxn matrices over a distributive lattice. We define

(L1) $A \gamma B = [a_{ij}] \gamma [b_{ij}] = [c_{ij}]$; $c_{ij} = a_{ij} \gamma b_{ij}$

$A \lambda B = [a_{ij}] \lambda [b_{ij}] = [c_{ij}]$; $c_{ij} = a_{ij} \lambda b_{ij}$

(L2) $A \triangle B = [a_{ij}] \triangle [b_{ij}] = [c_{ij}]$; $c_{ij} = \lambda (a_{ik} \gamma b_{kj})$

The following properties, most of which will be useful in the sequel, are derived immediately from the definitions:

(L3) Operations γ and λ on matrices satisfy commutative, associative, absorptive, distributive and idempotent laws similar to (1) 1. ~ 5.

(L4) Operation $\triangle$ on matrices satisfies an associative law.

(L5) $A \triangle (B \lambda C) = (A \triangle B) \lambda (A \triangle C)$,

$(A \lambda B) \triangle C = (A \triangle C) \lambda (B \triangle C)$

We call the algebra satisfying axioms (2) "Algebra β_1" tentatively. Let A and B be nxn matrices over Algebra β_1. We define

(A1) $A \vee B = [\ a_{ij}\ ] \vee [\ b_{ij}\ ] = [\ c_{ij}\ ]\ ;\ c_{ij} = a_{ij} \vee b_{ij}$

$A \wedge B = [\ a_{ij}\ ] \wedge [\ b_{ij}\ ] = [\ c_{ij}\ ]\ ;\ c_{ij} = a_{ij} \wedge b_{ij}$

(A2) $A \vartriangle B = [\ a_{ij}\ ] \vartriangle [\ b_{ij}\ ] = [\ c_{ij}\ ]\ ;\ c_{ij} = \bigwedge_{k=1}^{n} (a_{ik} \vee b_{kj})$

The following properties are derived from above definitions.

(A3) Operations V and Λ on matrices satisfy commutative, associative, and distributive laws. $A \wedge (A \vee B) = A.$

(A4) Operation $\vartriangle$ on matrices satisfies an associative law.

(A5) $A \vartriangle (B \wedge C) = (A \vartriangle B) \wedge (A \vartriangle C)$,

$(A \wedge B) \vartriangle C = (A \vartriangle C) \wedge (B \vartriangle C)$

Definition 1. A polynomial **f** is said to be the canonical form if it is written in the following form;

$$\mathbf{f} = (\overset{p}{\underset{i}{\gamma}}\ a_{\pi_i}) \lambda (\overset{r}{\underset{i}{\gamma}}\ a_{\rho_i}) \lambda \cdots \lambda (\overset{t}{\underset{i}{\gamma}}\ a_{\tau_i})$$

Definition 2. $\mathbf{a}_i$ is said to be the norm of $\mathbf{a}_i$ and denoted by $a_i = \| \mathbf{a}_i \|$. The norm of a polynomial **f** is defined as follows.

$$\| f \| = \| (\overset{a}{\underset{i}{\gamma}}\ \mathbf{a}_{\alpha_i}) \lambda (\overset{b}{\underset{i}{\gamma}}\ \mathbf{a}_{\beta_i}) \lambda \cdots \lambda (\overset{d}{\underset{i}{\gamma}}\ \mathbf{a}_{\delta_i}) \|$$
$$= (\overset{a}{\underset{i}{\vee}} a_{\alpha_i}) \wedge (\overset{b}{\underset{i}{\vee}} a_{\beta_i}) \wedge \cdots \wedge (\overset{d}{\underset{i}{\vee}} a_{\delta_i}) .$$

Here, it is assumed that $(\overset{a}{\underset{i}{\gamma}}\ \mathbf{a}_{\alpha_i})$, ⋯ , $(\overset{d}{\underset{i}{\gamma}}\ a_{\delta_i})$ are irreducible.

From the definition of norm Lemma 1 is easily derived.

Lemma 1. $\| \mathbf{f}_1 \lambda \mathbf{f}_2 \lambda \cdots \lambda \mathbf{f}_n \| = \| \mathbf{f}_1 \| \wedge \| \mathbf{f}_2 \| \wedge \cdots \wedge \| \mathbf{f}_n \|$.

In the case that all the diagonal elements of matrix $\mathbf{A}(A)$ are $\mathbf{a}_\circ$ ($a_\circ$), $\mathbf{A}(A)$ is denoted by $.\mathbf{A}(.A)$ or $[.\mathbf{a}_{ij}]([.a_{ij}])$. Let a polynomial $\mathbf{f}_{ij}$ have its element $\mathbf{a}_i$ replaced by a_i and the operations γ and λ by V and Λ , respectively, and let f_{ij} denote the resulting polinomial.

As it is well known, there is the following theorem with regards to a matrix over a distributive lattice.

Theorem 2[11]. $[.\mathbf{a}_{ij}]^{m\vartriangle} = [.\mathbf{a}_{ij}]^{(m+1)\vartriangle}$

where $[.\mathbf{a}_{ij}]$ is an nxn matrix and m is an integer not larger than n-1.

Lemma 3. If $\| \mathbf{f}_{ij} \| = f_{ij}$, then $\| [.\mathbf{f}_{ij}]^{m\vartriangle} \| = [\ \| .\mathbf{f}_{ij} \| \ ]^{m\vartriangle} = [.f_{ij}]^{m\vartriangle}$

Proof. $\| [.\mathbf{f}_{ij}]_{pq}^{m\vartriangle} \| = \| \underset{k_1,\cdots,k_{m-1}}{\lambda} (\mathbf{f}_{pk_1} \gamma \mathbf{f}_{k_1k_2} \gamma \cdots \gamma \mathbf{f}_{k_{m-1}q}) \|$

$= \underset{k_1,\cdots,k_{m-1}}{\bigwedge} (f_{pk_1} \vee f_{k_1k_2} \vee \cdots \vee f_{k_{m-1}q})$

$= [.f_{ij}]_{pq}^{m\vartriangle}.$

After this, we put $a_i \vee a_j = a_i + a_j$, $a_i \wedge a_j = max(a_i, a_j) = a_i\ T\ a_j$,

$a_\circ = 0$, *and* $a_\circ = \infty$.

3. Analysis of the Loop-Free Longest Path

Let d_{ij} denote the distance assigned to an element e_{ij} in a graph, and $\mathbf{d}_{ij}$ its respective composite quantity. Then the terminal loop-free longest distance matrix $[\ D_{ij}\ ]$ can be obtained by the following theorem.

Theorem 4. $[\ D_{ij}\ ] = \|\ [.\ \mathrm{d}ij\ ]^{m\triangle}\ \| = [\ \|.\ \mathrm{d}_{ij}^{m\triangle}\|\ ]$

Proof. For $[.\ \mathbf{d}_{ij}\ ]^{m\triangle}$ type operations the norm is defined in the form of irreducible path expressions[11]. Therefore, we can obtain the maximal sum of the distances of the elements which form a loop-free path.

Let $\mathbf{d}'_{ij}{}^{m\triangle}$ be i-j entry of $[\ d_{ij}\ ]^{m\triangle}$. If a graph is loop-free directed, $\mathbf{d}_{ij}{}^{m\triangle}$ is always loop-free without applying absorptive laws. Therefore, the relation $\|\ \mathbf{d}'_{ij}\ \| = d_{ij}$ holds. Hence, by Lemma 3 we get

Lemma 5. If a graph is loop-free directed, then $[\ D_{ij}\ ] = [.\ d_{ij}\ ]^{m\triangle}$.

We define the determinants $\|.\ \mathbf{d}_{ij}\ \|$ ($|.\ d_{ij}\ |$) of $[.\ \mathbf{d}_{ij}\ ]$ ($[.\ d_{ij}\ ]$) by

$$\|.\ \mathrm{d}_{ij}\ \| = \|\ \lambda\ (\ \mathrm{d}_{1h_1}\ \gamma\ \mathrm{d}_{2h_2}\ \gamma\ \cdots\ \gamma\ \mathrm{d}_{nh_n}\ )\ \|$$

$$|\ \mathrm{d}_{ij}\ | = \Lambda\ (\ \mathrm{d}_{\overline{1h_1}}\ \mathrm{V}\ \mathrm{d}_{2h_2}\ \mathrm{V}\ \cdots\ \mathrm{V}\ \mathrm{d}_{nh_n}\ )$$

where the operations λ and Λ are taken over all permutations $(h_1, h_2, \cdots, h_n)$ of(1, 2, ... , n). In order to find desired results by using determinants, we employ the following formula.

Theorem 6. $D_{ij} = \|{*D_{ij}}\ \|$.

where ${*D_{ij}}$ is the cofactor of $\mathbf{d}_{ji}$ in $\|.\ \mathbf{d}_{ij}\ \|$.

Lemma 7. In the case of a loop-free directed graph, $D_{ij} = {*D_{ij}}$.

where ${*D_{ij}}$ is the cofactor of d_{ji} in $|.\ d_{ij}\ |$.

Network-theoretical problems of communication systems have n variables polynomial representations similar to a boolean function in switching circuit theory, and also have the problem in this paper its polynomial representation.

Theorem 8. $D_{ij}\ (\ d_1,\ d_2,\ \cdots,\ d_n\ )$

$$= \{d_1 + D_{ij}\ (\ 0,\ d_2,\ \cdots,\ d_n\ )\}T\ D_{ij}\ (\ \infty,\ d_2,\ \cdots,\ d_n\ ).$$

Lemma 9. $D_{ij}\ (\ d_1,\ d_2,\ \cdots,\ d_n\ )$

$$= T\ \{D_{ij}\ (\ y_{\epsilon 1},\ y_{\epsilon 2},\ \cdots,\ y_{\epsilon n}\ ) + (\ \Sigma\ d_{k\epsilon k}\ )\}$$

where $d_{k\epsilon k} = d_k$ for $y_{\epsilon k} = 0$, and $d_{k\epsilon k} = 0$ for $y_{\epsilon k} = \infty$.

The loop-free longest paths can be found effectively by calculating the path expression of $\mathbf{d}_{ij}{}^{m\triangle}$ by means of diakoptical analysis[10] and then obtaining their norms. By Lemma 5, it is possible to carry out diakoptical method merely by operating on coefficients (distances) when a graph is loop-free directed.

Theorem 10 In the case of a loop-free directed graph,

$$[D_{ij}\ ] = [\ (\ \underset{\mu}{T}\ [\ .\tilde{\delta}^{m}_{\mu}]\ )T\ [.\tilde{\Delta}]\ ]^{m m\triangle}$$

$$[\ D_{ij}\ ] = (\tilde{\delta})G\bar{\nu}\ \triangle\ (.m_{d}^{m}m)\triangle(.\tilde{\delta})\bar{\nu}GT[.\tilde{\delta}\ ]$$

where the equations are metric expressions of corresponding equations in reference 10 and operations are carried out in accordance with Algebra β_1. Moreover, the subscript d attached to M is to indicate that the matrix is written in metric form.

Finding partial solutions by means of diakoptical analysis is similar to finding the entire solutions. In particular, when a graph is loop-free directed, it is possible to find partial solutions merely through operations on coefficients.

Theorem 11 In the case of a loop-free directed graph,

$$[D_{ij}]_{\bar{g}_\mu \bar{g}_\mu} = (\tilde{\delta}_\mu{}^{\tilde{m}_\mu})_{\tilde{g}_\mu \tilde{v}_\mu} \triangle (.md^{mm})_{\tilde{v}_\mu \tilde{v}_\nu} \triangle (.\tilde{\delta}_\nu{}^{\tilde{m}_\nu})_{\tilde{v}_\nu \bar{g}_\nu}$$

$$[D_{ij}]_{\bar{g}_\mu \bar{g}_\mu} = (.\tilde{\delta}_\mu{}^{m_\mu})_{\bar{g}_\mu v_\mu} \triangle (md^{mm})_{v_\mu v_\mu} \triangle (.\tilde{\delta}_\mu{}^{\tilde{m}_\mu})_{v_\mu g_\mu} T (.\tilde{\delta}_\mu{}^{\tilde{m}_\mu})_{\bar{g}_\mu \bar{g}_\mu}$$

where the equations are metric expressions of corresponding equations in reference 10 and the operations are carried out in accordance with Algebra β_1.

4. Conclusions

It is clear that, from mathematical points of view, the longest path problem may be regarded as a class of network-theoretical problems of communication systems (or combinatorial network problems). But, on the contrary to expectations, the longest path problem and the shortest path problem are neither dual nor isomorphic, mathematically. For instance, the applicable range of Lemma 5, Lemma 7, Theorem 10 and Theorem 11 is narrower than that of the corresponding formulae for the shortest path problem.

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