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In the calculation of the distortion produced in a transmission system of FM wave, the approximation so far made has considerable error in certain circumstance. In this paper, the amplitude and phase characteristics of the system are represented by a power series of frequency and some new theoretical formulas which have sufficient accuracy in the calculation of the distortion are derivated.

I. Introduction

With the advance of multiplex communication, the limitation for the distortion which occurs in the transmission system becomes extremely severe in comparison with the case of single channel.

The theoretical analysis of the distortion produced in a linear transmission system of FM waves have been made by many authors.[1)~5)] However, L.H.Enloe and C.L.Ruthroff[6)] pointed out that all of these procedures have considerable amount of error, because they use the simple relation (1) to determine the phase distortion $\theta_d(t)$. The meaning of (1) is as the following. Let the input signal of a transmission system be $S_i(t)$, the output be $S_0(t)$, and the phase distortion is represented as $\theta_d(t) = Im \log [S_0(t)/S_i(t)]$. When the distortion is small $S_0(t)$ and $S_i(t)$ are nearly equal, so $S_0(t)/S_i(t) \equiv 1+P(t)+jQ(t)$, and $|P(t)+jQ(t)|<1$, where $P(t)$ and $Q(t)$ are real. With this restriction the phase distortion can be written as follows:

$$\theta_d(t) \approx Q(t)\,. \tag{1}$$

For computing the distortion of the transmission system of low distortion the methods using the relation of (1) will certainly be inadequate. From this point of view, we induced the theoretical fromulas that hacve sufficient accuracy in computing the distortion of any system when the amplitude and the phase characteristics of the system are given. The formulas was induced under the concept of the instantaneous frequency given by A. Ditl.[7)] The instantaneous angular frequency $d\theta(t)/dt$ is defined as $d\theta(t)/dt = \mathrm{I}m\,[1/S_0(t) \cdot dS_0(t)/dt]$. When FM wave with the peak frequency deviation $\varDelta\omega$ is applied to a linear system, the instantaneous angular frequency is represented by the sum of the carrier angular frequency ω_c and the transent term $Re[\sigma(t)]$. This transient term is expressed by a power series of $\varDelta\omega$ as follows:

$$Re\,[\sigma(t)] = \sum_{n=1}^{M} Re\,[W_n]\cdot \varDelta\omega^n\,. \tag{2}$$

The harmonic components are determined from $Re[W_n]$ which are independent of

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the peak frequency deviation. The first terms of W_1, W_2, $W_3 \cdots$ derived from Eq. (2) is identical with the result obtained by using Carson and Fry technigue[8] on Eq. (1) and expanding the transfer fanction in a power series of the frequency. However, the harmonic distortion calculated by using only the first terms is not sufficiently accurate as illustrated in the example of Chapter 4. Moreover, the error has the larger value when only the first terms are used on a system having unsymmetry about the carrier frequency. The Eq. (2) has not any restriction such as made to (1), so the formulas obtained in this paper can be expected to have sufficient accuracy.

2. Derivation of theoretical formulas

Let the FM wave $S_i(t)=\exp j[\omega_c t+\Delta\omega\int^t f(t)dt]$ be applied to a linear transmission system with the transmission charcateristic $T(j\omega)$. The output wave can be represented by complex form as $S_0(t)=G(t)\exp[j\theta(t)]$, where $f(t)$ is the modulating signal, $G(t)$ and $\theta(t)$ and $\theta(t)$ are the instantaneous amplitude and phase angle, respectivly.

The instantaneous angular frequency $d\theta(t)/dt$ is defined as follows:

$$\theta^{(1)}(t) = I_m[S_0^{(1)}(t)/S_0(t)]\,. \tag{3}$$

Where $Im\,[X]$ means the imaginary part of X.

Now, let the frequency spectrum of $S_i(t)$ be $F_i(j\omega)$, and the spectrum $F_0(j\omega)$ of the output wave of the system is given by $F_0(j\omega)=T(j\omega)\cdot F_i(j\omega)$, so the output wave $S_0(t)$ is obtained by performing the inverse Fourier transform of $F_0(j\omega)$. Therefore, (3) can be written as follows:

$$\theta^{(1)}(t)=I_m\left[\int_{-\infty}^{\infty} j\omega\cdot F_i(j\omega)\cdot T(j\omega)\cdot\exp[j\omega t]\,d\omega\Big/\int_{-\infty}^{\infty} F_i(j\omega)\cdot T(j\omega)\exp[j\omega t]d\omega\right]. \tag{4}$$

When the Fourier transform of $\exp\left[j\Delta\omega\int^t f(t)dt\right]$ is $F(j\omega)$, $F_i(j\omega)$ is able to express as $F(j\omega-j\omega_c)$. Now, we introduce the variable $x=\omega-\omega_c$. Substituting the normalized transmission characteristic $T_N(j\omega)$ for $T(j\omega)$, we expand it in the power series of x as follows:

$$\left.\begin{aligned} T_N(j\omega) &= A_N(\omega)\exp[j\Phi_N(\omega)] = \left(\sum_{n=0}^{\infty}\frac{A_N^{(n)}(0)}{n!}x^n\right)\cdot\exp\left[j\sum_{n=2}^{\infty}\frac{\Phi_N^{(n)}(0)}{n!}x^n\right] \\ &= \sum_{n=0}^{\infty}\alpha_n\cdot x^n \end{aligned}\right\} \tag{5}$$

The linear phase term in $\Phi_N(j\omega)$ has no influence upon the distortion, so it subtracted out before performing calculation. Thus, (4) is expressed as follows:

$$\left.\begin{aligned} \theta^{(1)}(t) &= \omega_c+Re\left[\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\alpha_n x^{n+1}F(jx)\exp[jxt]dx\Big/\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\alpha_n x^n F(jx)\exp[jxt]dx\right] \\ &= \omega_c+Re[\sigma(t)] \end{aligned}\right\} \tag{6}$$

Where $Re[Y]$ means the real part of Y, $Re[\sigma(t)]$ is the transient term, and α_n is the constant determined by Eq. (5). As we are interested in only $Re[\sigma(t)]$, so the calculation will be made only about $\sigma(t)$ in the following.

By performing the inverse Fourier transform of Eq. (5), the denominator and numerator can be expressed by a power series of $\Delta\omega$ as follows:

$$\sigma(t) = \sum_{k=1}^{\infty} U'_{k,n}\cdot X_{k-1,n}\cdot \Delta\omega^k / 1 + \sum_{k=1}^{\infty} U'_{k,n}\cdot X_{k,n}\cdot \Delta\omega^k \,, \tag{7}$$

where $U_{k,n}$ and $X_{k,n}$ and the column matrices decided from $f(t)$ and α_n, respectively. The $U'_{k,n}$ means a transposed matrix of $U_{k,n}$ and the suffix n indicates the n row elements of a column matrix.

$$U_{1,n} = \begin{pmatrix} f \\ f^{(1)} \\ f^{(2)} \\ \vdots \\ f^{(n-1)} \end{pmatrix},\quad U_{2,n} = \begin{pmatrix} f^2 \\ (f^2)^{(1)} + ff^{(1)} \\ (f^2)^{(2)} + (ff^{(1)})^{(1)} + ff^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (f^2)^{(n-1)} + (ff^{(1)})^{(n-2)} + \cdots + ff^{(n-1)} \end{pmatrix}, \cdots$$

$$U_{k,n} = \begin{pmatrix} u_{k,1} \\ u_{k,2} \\ u_{k,3} \\ \vdots \\ u_{k,n} \end{pmatrix} = \begin{pmatrix} u_{k,1} \\ u_{k,1}^{(1)} + f\cdot u_{k-1,2} \\ u_{k,2}^{(1)} + f u_{k-1,3} \\ \cdots\cdots\cdots\cdots \\ u_{k,n-1}^{(1)} + f u_{k-1,n} \end{pmatrix}, \tag{8}$$

$$X_{0,n} = \begin{pmatrix} \alpha_0 \\ -j\alpha_1 \\ -\alpha_2 \\ \vdots \\ (-j)^{n-1}\cdot\alpha_{n-1} \end{pmatrix},\quad X_{1,n} = \begin{pmatrix} \alpha_1 \\ -j\alpha_2 \\ -\alpha_3 \\ \vdots \\ (-j)^n\alpha_n \end{pmatrix}, \cdots X_{k,n} = \begin{pmatrix} \alpha_k \\ -j\alpha_{k+1} \\ -\alpha_{k+2} \\ \vdots \\ (-j)^{n-1}\cdot\alpha_{k+n-1} \end{pmatrix}. \tag{9}$$

Where $u_{k,i}$ $(i=1,2,3\cdots)$ is the element of $U_{k,n}$. Moreover, by using $V_{k,k-1}=U'_{k,n}\cdot X_{k-1,n}$ and $V_{k,k}=U'_{k,n}X_{k,n}$ on Eq. (7), we have the following equation.

$$Re[\sigma(t)] = Re[\sum_{k=1}^{\infty} V_{k,k-1}\cdot\Delta\omega^k / 1 + \sum_{k=1}^{\infty} V_{k,k}\Delta\omega^k] = \sum_{k=1}^{\infty} Re[W_k]\cdot\Delta\omega^k\,, \tag{10}$$

where W_k are as follows:

$$\left.\begin{aligned} W_1 &= V_{1,0} \\ W_2 &= V_{2,1} - W_1\cdot V_{1,1} \\ W_3 &= V_{3,2} - W_2\cdot V_{1,1} - W_1\cdot V_{2,2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ W_k &= V_{k,k-1} - W_{k-1}\cdot V_{1,1} - W_{k-2}\cdot V_{2,2} \cdots W_1\cdot V_{k-1,k-1}\,. \end{aligned}\right\} \tag{11}$$

As mentioned above, when the amplitude and the phase characteristics of a system are given, the instantaneous angular frequency of FM wave modulated by a signal $f(t)$, *i.e.*, the wave demodulated by the ideal discriminator with the ideal limiter can be computed from Eqs. (5)~(11). In general, W_k has the higher order terms of α_i and the product terms of a_i, a_j, a_k, $\cdots$. However, we can make the reasonable approximation from Eq. (10) by considering the signal $f(t)$, the constant a_i and the deviation $\Delta\omega$.

3. Harmonic distortion

Let the modulation be a sinusoidal wave. Then $f(t)=\cos pt$, where p is the angular frequency of modulation. Substituting $\cos pt$ for $f(t)$ in Eq. (8), from Eqs. (10) and (11) we obtain the following equations, provided $\alpha_n p^n=\beta_n$,

$$\left.\begin{aligned} W_1 &= A_1 \cos pt + jB_1 \sin pt \\ A_1 &= \beta_0+\beta_1+\beta_4+\cdots = \sum_{i=0}^{\infty} \beta_{2i} \\ B_1 &= \beta_1+\beta_3+\beta_5+\cdots = \sum_{i=0}^{\infty} \beta_{2i+1} \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} W_2 &= \frac{1}{2p}\{A_2 \cos 2pt + jB_2 \sin 2pt\} \\ A_2 &= 6\beta_3+30\beta_5+126\beta_7+\cdots-2\sum_i\sum_j \beta_{2i+1}\beta_{2j+2} \\ B_2 &= 2\beta_2+14\beta_4+62\beta_6+\cdots-\sum_i\sum_j(\beta_{2i+2}\beta_{2j+2}+\beta_{2i+1}\beta_{2j+1}) \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} W_3 &= \frac{1}{4p^2}\{A_3 \cos 3pt + jB_3 \sin 3pt + A_3' \cos pt + jB_3' \sin pt\} \\ A_3 &= 18\beta_4+270\beta_6+\cdots-\sum_i \beta_{2i+2}(4\beta_2+22\beta_4+94\beta_6+\cdots) \\ &\quad -\sum_j \beta_{2i+1}(9\beta_3+45\beta_5+189\beta_7+\cdots) \\ &\quad -\sum_i\sum_j\sum_k(\beta_{2i+2}\beta_{2j+2}\beta_{2k+2}+3\beta_{2i+1}\beta_{2j+1}\beta_{2k+2}) \\ B_3 &= 3\beta_3+75\beta_5+903\beta_7+\cdots-\sum_i \beta_{2i+2}(9\beta_3+45\beta_5+189\beta_7+\cdots) \\ &\quad -\sum_i \beta_{2i+1}(4\beta_2+22\beta_4+94\beta_6+\cdots) \\ &\quad -\sum_i\sum_j\sum_k(\beta_{2i+1}\beta_{2j+1}\beta_{2k+1}+3\beta_{2i+2}\beta_{2j+2}\beta_{2k+1}) \\ A_3' &= 5\beta_4+29\beta_6+\cdots+\sum_i \beta_{2i+2}(6\beta_4+30\beta_6+\cdots) \\ &\quad -\sum_i \beta_{2i+1}(3\beta_3+15\beta_5+113\beta_7+\cdots) \\ &\quad -\sum_i\sum_j\sum_k(\beta_{2i+2}\beta_{2j+2}\beta_{2k+2}-\beta_{2i+1}\beta_{2j+1}\beta_{2k+2}) \\ B_3' &= 3\beta_3+14\beta_5+62\beta_7+\cdots+\sum_i \beta_{2i+2}(3\beta_3+15\beta_7+63\beta_7+\cdots) \\ &\quad -\sum_i \beta_{2i+1}(4\beta_2+10\beta_4+34\beta_6+\cdots) \\ &\quad +\sum_i\sum_j\sum_k(\beta_{2i+1}\beta_{2j+1}\beta_{2k+1}-\beta_{2i+2}\beta_{2j+2}\beta_{2k+1}). \end{aligned}\right\} \tag{14}$$

The major terms are limited to the third term W_3 in the above formulas, but W_4, W_5, $\cdots$, will be obtained by the same procedures. As expressed in (12), (13) and (14), we are able to derived the fundamental component from W_1, W_3, W_5, $\cdots$, the second hormonic from W_2, W_4, $\cdots$, and the third harmonic from W_3, W_5, $\cdots$. When the amplitude and the phaes characteristics of a system have even symmetry and odd symmetry about the carrier frequency, respectively, the even terms of W_k disappear. When a system has the linear phase and its amplitude is symmetrical ahout carrier frequency, W_1, W_3, W_5, $\cdots$ are represented by cosine terms. And when a system has the flat amplitude but odd symmetry of phase, W_1, W_3, W_5, $\cdots$ are represented by sine and cosine terms.

Now, let W_1, W_2, W_3, $\cdots$ be approximated by only the first term in (12) and (13), we have

$$\left.\begin{aligned} W_1 &\approx \cos pt \\ W_2 &\approx \frac{1}{2p}(6\beta_3 \cos 2pt + j2\beta_2 \sin 2pt) \\ W_3 &\approx \frac{1}{4p^2}(18\beta_4 \cos 3pt + j3\beta_3 \;\text{ in } 3pt + 5\beta_4 \cos pt + j3\beta_3 \sin pt) \end{aligned}\right\} \tag{15}$$

The above equations are identical with the results obtained by using Carson and Fry technique on (1). β_1 has no relation to the distortion in Eq. (15), however β_1, β_2, β_3, $\cdots$ are contained in the higher order harmonic components obtained from Eqs. (12)~(14) as follows:

$$\left.\begin{aligned} &\text{fundamental amplitude:} \\ &\qquad \Delta\omega\left\{\left(Re[A_1] + \frac{\Delta\omega^2}{4p^2} Re[A_3']\right)^2 + \left(I_m[B_1] + \frac{\Delta\omega^2}{4p^2} I_m[B_3']\right)^2\right\}^{1/2} \\ &\text{the econd harmonic amplitude:} \\ &\qquad \frac{\Delta\omega^2}{2p}\{(Re[A_2])^2 + (I_m[B_2])^2\}^{1/2} \\ &\text{the third harmonic amplitude:} \\ &\qquad \frac{\Delta\omega^3}{4p^2}\{(Re[A_3])^2 + (I_m[B_3])^2\}^{1/2}\,. \end{aligned}\right\} \tag{16}$$

Where, we supposed that the harmonic components obtained from $Re[W_4]\cdot\Delta\omega^4$, $Re[W_5]\cdot\Delta\omega^5$) $\cdots$ are negligibly small.

4. Examples

In general, when the transmission charactlistic is given, we can at once calculate the harmonic distortions from (16) by obtaining β_i (i=0,1,2,3,$\cdots$) of (12)~(14). As examples of calculation we show the cases of [1] the m-stages synchronized tuned amplifier and [2] the m-stages critical coupled double tuned amplifier as a band pass amplifier. The results of the calculation are shown in Table I and II. In Table I β_i of each case are listed. In the Table II the klirr-attenuations of the third harmonic are shown for the case which the peak frequency deviation is 75KC, modulating frequency is 20 KC and the total bandwidth B_m is twice the sum of them. Approximations are obatined from Eq. (15).

The result of the example [1] shows that there are the differences of considerable amounts between the value obtained from Eq. (14) and that of the approximation obtained from Eq. (15).

Table I. Characteristics and coefficients in [1] and [2], provided $\beta_n = a_n p^n$, m=no. of stages.

		[1] *m-stages synchronized tuned amplifier*	[2] *m-stages critical coupled double tuned amplifier*
Amplitude characteristic $A_N(x)$		$\left\{1+\left(\frac{x}{\pi B}\right)^2\right\}^{-m/2}$	$\left\{1+\left(\frac{x}{\pi B}\right)^4\right\}^{-m/2}$
Phase characteristic $\Phi_N(x)$		$-m \arctan \frac{x}{\pi B}$	$-m \arctan \frac{\sqrt{2}x/\pi B}{1-(x/\pi B)^2}$
Bandwidth per a stage B		$B_m\ (2^{1/m}-1)^{-1/2}$	$B_m\ (2^{1/m}-1)^{-1/4}$
β_0	$A_{N(0)}$	1	1
β_1	$A^{(1)}_{N(0)}$	—	—
β_2	$\frac{A^{(2)}_{N(0)}}{2!}p^2$	$-\frac{m}{2}\left(\frac{p}{\pi B}\right)^2$	—
β_3	$j\frac{\Phi^{(3)}_{N(0)}}{3!}p^3$	$j\frac{m}{3}\left(\frac{p}{\pi B}\right)^3$	$-j\frac{\sqrt{2}m}{3}\left(\frac{p}{\pi B}\right)^3$
β_4	$\frac{A^{(4)}_{N(0)}}{4!}p^4$	$\frac{m(m+2)}{8}\left(\frac{p}{\pi B}\right)^4$	$-\frac{m}{2}\left(\frac{p}{\pi B}\right)^4$
β_5	$j\left(\frac{\Phi^{(5)}_{N(0)}}{5!}+\frac{A^{(2)}_{N(0)}\Phi^{(3)}_{N(0)}}{2\cdot 3!}\right)p^5$	$-j\frac{m(5m+6)}{30}\left(\frac{p}{\pi B}\right)^5$	$j\frac{\sqrt{2}m}{5}\left(\frac{p}{\pi B}\right)^5$
β_6	$\frac{A^{(6)}_{N(0)}}{6!}p^6$	$-\frac{2m(m+2)(m+4)}{5!}\left(\frac{p}{\pi B}\right)^6$	—

Table II. Klirr-attenuations calculated from Table I.

	[1]			[2]		
No. of stages	1	2	3	1	2	3
Third harmonic klirr-attennation (db)	33,44	37,06	38,86	23,80	22,46	21,28
Third harmonic klirr-attenuation by approximation (db)	25,94	31,24	34,94	23,02	21,60	20,48

5. Conclusion

The amplitude and the phase characteristics of a transmission system are expended by a power series, and the instantaneous frequency of FM wave is calculated from Eqs. (6),(10) and (11). As shown in Eqs. (12)~(14), the higher order and the product terms of β_i, β_j, β_k, $\cdots$, which are not found in the approximate method, appear in this accurate calculation of the distortion. Therefore, when the higher order derivatives of the transmission characteristics can not be negligible due to unsymmetry etc., we can obtain the sufficiently accurate values of the harmonic distortions from Eqs. (12) (14) and (16). Thus, we believe that the foumulas derived in this paper have the more accuracy than the approximation so far made.

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