Foundations for the Boundary Problems of the Polyphase Transmission Lines Considering the Initial Conditions I

Foundations for the Boundary Problems of the Polyphase Transmission Lines Considering the Initial Conditions-I

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This report presents the theoretical analyses for the transient problems of the polyphase transmission system, considering both initial conditions and boundary conditions.

I Introduction

The transient problems of the polyphase transmission system have been studied theoretically by several authors, ignoring either the initial conditions or boundary conditions(1)(2). It was not possible to estimate the line potentials by previous methods taking into account of these conditions simultaneously.

The present report deals with such boundary problems by a newly-established analytical method. The essential point of the new method are summarized as : diagonal transformations of the "Line Equations" reduce the multi-conductor system to the group of independent lines whose mutual effects due to traveling waves along the other lines may be neglected ; Green functions which are used satisfy the boundary conditions of these independent lines.

For the purpose of explaining the foundations of this method, it is most convenient to assume that the system is grounded through electrical sources at both ends of the transmission lines. In more general cases, for example lines closed by impedances or admittances, the difficulty and complexity of solutions will increase considerably. Nevertheless the results for such cases will be reported in this paper.

II Analytical Method for a Multi-conductor System in Matrix Forms

If the matrices $[E]$ and $[I]$ denote the voltages and currents of the point x along the transmission lines, respectively, the differential equations(1)(2) of n-phase system are, as is well-known, expressed by the following matrix forms

$$\left.\begin{aligned} -\frac{\partial [E]}{\partial x} &= [L]\frac{\partial}{\partial t}[I]+[R]\,[I] \\ -\frac{\partial [I]}{\partial x} &= [C]\frac{\partial}{\partial t}[E]+[G]\,[E] \end{aligned}\right\} \tag{1}$$

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where

$$[E]=\{E_1,E_2,\cdots\cdots\cdots, E_{n-1},E_n\}$$

$$[I]=\{I_1,I_2,\cdots\cdots\cdots, I_{n-1},I_n\}$$

$$[R]=\begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & \ddots & \vdots \\ R_{n1} & \cdots & R_{nn} \end{bmatrix}, \quad [L]=\begin{bmatrix} L_{11} & \cdots & L_{1n} \\ \vdots & \ddots & \vdots \\ L_{n1} & \cdots & L_{nn} \end{bmatrix}$$

$$[C]=\begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix}, \quad [G]=\begin{bmatrix} G_{11} & \cdots & G_{1n} \\ \vdots & \ddots & \vdots \\ G_{n1} & \cdots & G_{nn} \end{bmatrix}$$

$[R]$, $[L]$, $[C]$ and $[G]$ denote nth degree square matrices whose elements are composed of resistances, inductances, capacitances and leakances of the lines per unit length, with the effect of the ground return taken into account simultaneously.

Substituting the Heaviside operator $s=\partial/\partial t$ for the time derivative, (1) becomes

$$\left.\begin{aligned} -\frac{d[e]}{dx}&=\{s[L]+[R]\}[i]-s[L][I_{t=0}] \\ -\frac{d[i]}{dx}&=\{s[C]+[G]\}[e]-s[C][E_{t=0}] \end{aligned}\right\} \qquad (2)$$

Now consider the case, in which the lines are closed by general impedances consisting of any inductances, capacitances and resistances, through the electrical sources. One of the most general types of this kind has line terminals at which there are shunt admittances $[Y_0]$, $[Y_e]$ to ground ; and n lines joined to e. m. f. s through series impedances $[Z_0]$, $[Z_e]$, respectively. Such a system is shown in Fig. 1.

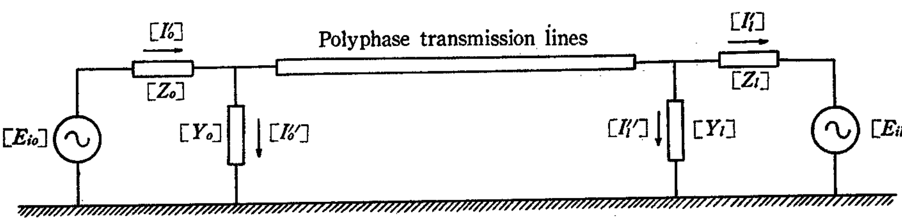


Fig. 1 Simplified equivalent circuit of polyphase transmission system.

Suppose that $[I_0']$ and $[I_0'']$ represent currents flowing in the series impedances $[Z_0]$ and shunt admittances $[Y_0]$ at the sending end $x=0$, respectively. Then the boundary conditions at $x=0$ are given by

$$\left.\begin{aligned}&[I_0'] = ([I])_{x=0} + [I_0''] \;, \qquad [I_0''] = [Y_0]\,([E])_{x=0}\\ &[E_{i0}] = [Z_0]\,[I_0'] + ([E])_{x=0}\end{aligned}\right\} \tag{3}$$

where

$$[I_0'] = \{I_{0,1}',\ I_{0,2}',\ \cdots\cdots,\ I_{0,n}'\},\quad [I_0''] = \{I_{0,1}'',\ I_{0,2}'',\cdots\cdots,\ I_{0,n}''\}$$

$$[E_{i0}] = \{E_{i0,1},\ E_{i0,2},\ \cdots\cdots,\ E_{i0,n}\}$$

$[Z_0]$ and $[Y_0]$ must be square and of order n.

Furthermore, assuming that $[E_{t=0}]$ and $[I_{t=0}]$ denote the initial distributions of voltages and currents along the lines respectively, and making use of (3), we can readily obtain the relations, in operational form,

$$\begin{aligned}[i_{x=0}] = [sL_0+R_0]^{-1}\{[e_{i0}] + s\,[L_0]\,[\underset{t=0}{I_0'}]\} + s\,[C_0]\,[\underset{t=0}{E_{x=0}}]\\ -\{[sL_0+R_0]^{-1} + [sC_0+G_0]\}\,[e_{x=0}]\end{aligned} \tag{4}$$

in which $[sL_0+R_0]$ and $[sC_0+G_0]$ mean the s-functions of $[Z_0]$ and $[Y_0]$, respectively.

Substituting the first equality of (2) in (4) and using the transformation of

$$\left.\begin{aligned}&[e'] = [e] - [e_i]\\ &[e_i] = \left(1-\frac{x}{l}\right)\Big\{[U] + [sL_0+R_0]\;[sC_0+G_0]\Big\}^{-1}[e_{i0}]\\ &\qquad\qquad + \frac{x}{l}\Big\{[U] + [sL_e+R_e]\;[sC_e+G_e]\Big\}^{-1}[e_{ie}]\end{aligned}\right\} \tag{5}$$

the boundary conditions at $x=0$ reduce to

$$\left(\frac{d[e']}{dx}\right)_{x=0} + [a]\,([e'])_{x=0} = [e_i'] + [e_f] =: [e_f'] \tag{6}$$

where

$$[a] = -\,[sL+R]\Big\{[sL_0+R_0]^{-1} + [sC_0+G_0]\Big\} \tag{7}$$

$$[e_f] = s\Big\{[L]\,[\underset{t=0}{I_{x=0}}] - [sL+R]\Big([sL_0+R_0]^{-1}[L_0]\,[\underset{t=0}{I_0'}] + [C_0]\,[\underset{t=0}{E_{x=0}}]\Big)\Big\} \tag{8}$$

$$[e_i'] = \frac{1}{l}\Big\{[U] + [sL_0+R_0]\;[sC_0+G_0]\Big\}^{-1}[e_{i0}] - \frac{1}{l}\Big\{[U] + [sL_e+R_e]\;[sC_e+G_e]\Big\}^{-1}[e_{ie}] \tag{9}$$

The relations of the same type with (6) also hold good for the boundary conditions of receiving end $x=l$. There are

$$\left(\frac{d[e']}{dx}\right)_{x=l} + [b]\Big([e_i']\Big)_{x=l} = [e_i'] + [e_g] = [e_g'] \tag{10}$$

where

$$[b] = [sL+R]\Big\{[sL_e+R_e]^{-1} + [sC_e+G_e]\Big\} \tag{11}$$

$$[e_g] = s\left\{[L]\,[I_{\substack{x=e\\t=0}}] - [sL+R]\left([sL_e+R_e]^{-1}[L_e]\,[I'_e]_{t=0} - [C_e]\,[E_{\substack{x=e\\t=0}}]\right)\right\} \tag{12}$$

in which $[sL_e+R_e]$ and $[sC_e+G_e]$ are the operational forms of $[Z_e]$ and $[Y_e]$, respectively.

Differentiating the first equality of (2) with respect to x, substituting the second equality in it and making use of the transformation of (5), the operational equations to determine the line voltages are simplified to

$$\frac{d^2[e']}{dx^2} - [k]^2[e'] = [k]^2[e_i] + [Q] \tag{13}$$

in which

$$\left.\begin{aligned} [Q] &= s\left\{[L]\frac{d}{dx}[I_{t=0}] - [sL+R]\,[C]\,[E_{t=0}]\right\} \\ [k]^2 &= [sL+R]\,[sC+G] \end{aligned}\right\} \tag{14}$$

(13) are the general equations, whose solutions, subject to the boundary conditions, for example (6) and (10) yield the explicit equations of the transmission line transients.

In order to get the solutions of (13), the matrices involved in it must be transformed into diagonal matrices(3). We start with the existence of the transformation matrix $[\mu]$ and the diagonal matrix $[q]^2$ which is related by

$$[\mu]^{-1}[k]^2[\mu] = [q]^2 \tag{15}$$

Then, putting

$$[\mu]^{-1}[e'] = [e''] \tag{16}$$

(6), (10) and (13) become, upon postmultiplying by $[\mu]^{-1}$

$$\frac{d^2[e'']}{dx^2} - [q]^2[e''] = [\mu]^{-1}\left\{[k]^2[e_e] + [Q]\right\} \tag{17}$$

$$\left(\frac{d[e'']}{dx}\right)_{x=0} + [a']\left([e'']\right)_{x=0} = [\mu]^{-1}[e'_f],\ \left(\frac{d[e'']}{dx}\right)_{x=l} + [b']\left([e'']\right)_{x=l} = [\mu]^{-1}[e'_g], \tag{18}$$

where

$$[a'] = [\mu]^{-1}[a]\,[\mu],\quad [b'] = [\mu]^{-1}[b]\,[\mu]$$

A very interesting observation can be made at the point of a diagonal matrix $[q]^2$ in above expressions : (17) shows an equivalent n-phase lines, that are electrically independent of each other, with the both ends terminated through impedances. This is certainly desired considering the large degree of simplification that will result from the fact that no coupling exists between n-phase.

On the other hand. according to (15).

$$[\mu][q]^2[\mu]^{-1}=[k]^2$$

Therefore, the elements of $[q]^2$, i.e. q_r^2's, must satisfy the following characteristic equation :

$$\det\{q^2[U]-[k]^2\}=0 \tag{19}$$

in which $[U]$ denotes the unit matrix of order n. In other words, q_r^2's $(r=1, 2, \cdots\cdots, n)$ should be the latent roots of $[k]^2$.

Now, to have complete solutions, it is necessary to find solutions which satisfy the differential equations of (17) and boundary conditions of (18). One can readily check that the solutions of (17) are

$$\sinh[q_r]x\cdot[A']+\cosh[q_r]x\cdot[B']+[G_r(x,\xi)][C']$$

in which $[G_r(x,\xi)[$ is, in the form of Green function,

$$[G_r(x,\xi)]=\pm\frac{1}{2}[q_r]^{-1}\sinh[q_r](x-\xi), \quad x\gtreqless\xi$$

Hence, the solutions for the multi-conductor system are immediately obtained, if $[A'],[B']$ and $[C']$ can be determined so that the terminal conditions of the lines desire.[4]

Thus, the complete solutions may be written

$$\begin{aligned}
[e]=[e_i]&+\sum_{r=1}^{n}[K(q_r^2)]\Big[\sinh q_r x\Big\{[a]^{-1}[k]-[\varkappa_1]^{-1}[\varkappa_2]\Big\}^{-1}\Big\{[a]^{-1}[e_f']\\
&-[\varkappa_1]^{-1}[e_g']-\frac{1}{2}\sum_{r=1}^{n}\int_0^l\Big(\frac{1}{q_r}\sinh q_r\xi[U]-\cosh q_r\xi[a]^{-1}-[\varkappa_1]^{-1}[\nu]\Big)[K(q_r^2)][\psi]d\xi\Big\}\\
&+\cosh q_r x\Big\{[k]^{-1}[a]-[\varkappa_2]^{-1}[\psi_1]\Big\}^{-1}\Big\{[k]^{-1}[e_f']-[\varkappa_2]^{-1}[e_g']\\
&-\frac{1}{2}\sum_{r=1}^{n}\int_0^l\Big(\frac{1}{q_r^2}\sinh q_r\xi[a]-\frac{1}{q_r}\cosh q_r\xi[U]-[\varkappa_2]^{-1}[\nu]\Big)[K(q_r^2)][\psi]d\xi\Big\}\\
&+\frac{1}{2q_r}\Big\{\int_0^x\sinh q_r(x-\xi)[\psi]d\xi+\int_l^x\sinh q_r(\xi-x)[\psi]d\xi\Big\}\Big]
\end{aligned} \tag{20}$$

in which

$$[K(q_r^2)]=\frac{\prod\limits_{\substack{m=1,2,\cdots,n\\ m\neq r}}(q_m^2[U]-[k]^2)}{\prod\limits_{\substack{m=1,2,\cdots,n\\ m\neq r}}(q_m^2-q_r^2)} \tag{21}$$

$$\left.\begin{aligned}
[\varkappa_1]&=\sum_{r=1}^{n}(q_r\sinh q_r l[U]+\cosh q_r l[b])[K(q_r^2)]\\
[\varkappa_2]&=\sum_{r=1}^{n}(q_r\cosh q_r l[U]+\sinh q_r l[b])[K(q_r^2)]
\end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned}
[\psi]&=q_r^2[e_i(\xi)]+[Q(\xi)]\\
[\nu]&=\frac{1}{q_r}\sinh q_r(l-\xi)[b]+\cosh q_r(l-\xi)[U]
\end{aligned}\right\} \tag{23}$$

Hereupon, putting $[Z_0]=[Z_e]=[0]$, $[Y_0]=[Y_e]=[0]$ in above expressions, we have the operational solutions in the case that the both terminals are grounded to the earth, through electrical sources.

$$
\begin{aligned}
[e]=&\left(1-\frac{x}{l}\right)[e_{i0}]+\frac{x}{l}[e_{ie}]-\sum_{r=1}^{n}[K(q_r^2)]\Bigg(\left\{1-\frac{x}{l}\right.\\
&\left.-\frac{\sinh q_r(l-x)}{\sinh q_r l}\right\}[e_{i0}]+\left\{\frac{x}{l}-\frac{\sinh q_r x}{\sinh q_r l}\right\}[e_{ie}]\\
&+\frac{\sinh q_r(l-x)}{q_r\sinh q_r l}\int_0^x \sinh q_r\xi[Q(\xi)]d\xi+\frac{\sinh q_r x}{q_r\sinh q_r l}\int_x^l \sinh q_r(l-\xi)[Q(\xi)]d\xi\Bigg)
\end{aligned}
\tag{24}
$$

The potentials along the lines will be easily estimated by utilizing the inverse Laplace transformations for (20) or (24).

The operational expressions for the line currents may be derived by combining the second equality of (2) with (20) or (24).

Although there is no limit to the complexity of the impedance networks at both ends on the multi-conductor circuit, yet for most practical cases that shown in Fig. 1 is sufficiently general. Indeed, the procedure followed in setting up and solving the equations for the waves, as well as for the currents and voltages in all branches of any transition point network, is the same, so that the method of solution which was given can be applied generally.

III Method of Numerical Calculations Considering the Line Losses in Dissymmetrical System

The conventional treatment of transmission-line transients must be based on consideration of no-loss lines and symmetrical system. However, there are many important problems, in which line losses can not be ignored, such as in the theory of "Long Transmission Lines". Sometimes their influence is so vital as to change the characteristics of the phenomena, and erronous results are obtained if they are not considered. If we do not mind the trouble of complexity in calculations, mathematical solutions can be found from (20), taking into account of loss factors of conductors in dissymmetrical system.

But, it is more convenient to use a following method so as to avoid complicated mathematical calculations in order to get the numerical solutions.

(1) As an example of (20), let it be required to find line parameters, that is self-inductances, mutual-inductances, self-capacitances and etc., initial distributions, i. e. $[E_{t=0}]$, $[I_{t=0}]$, terminated impedances and admittances.

In the neighbourhood of the wave front, we can assume that s should become infinitely great. Then, substituting the line parameters in (19), q_r of the characteristic equation takes the form

$$q_r=\alpha_r s+\beta_r$$

in which α_r and β_r should be constants.

Herefrom, $[K(q_r^2)]$ will be readily founded, by using (21) and (14),

(2) From the values of the terminated impedances and admittances, the matrices $[e_i]$, $[e_i']$, $[a]$ and $[b]$ will be determined from (5), (9), (7) and (11), respectively.

(3) The matrices $[e_f]$, $[e_g]$ and $[Q]$ will be decided from (8), (12) and (14).

(4) Thus, we can find the forms of $[\varkappa_1]$, $[\varkappa_2]$, $[\psi]$ and $[\nu]$ by substituting q_r, $[e_i]$, $[b]$ and $[Q]$ in Equations (22) and (23).

(5) Now, insert q_r, $[K(q_r^2)]$, $[e_f]$, $[e_g]$, $[e_i]$, $[e_i']$, $[\varkappa_1]$, $[\varkappa_2]$, $[\psi]$, $[\nu]$ and $[Q]$ in (20), noticing the relations of (6) and (10), and integrate with respect to ξ.

(6) Then the line potentials at any time may be readily calculated by inverse transformation of (20).

(7) Line currents are also founded, if needed, from the first equality of (2).

Of course, in a case of this kind much time is saved by substituting the numerical values of line constants directly, rather than reducing the general equations.

The method in this chapter holds rigorously only for small values of t, but under actual conditions electrostatic transients are usually over within a fraction of a millisecond. In above method, it should be only required to evaluate the coefficients of operator "s" with different orders. Thus, this method would no doubt be of much use from an engineering point of view.

IV Numerical Examples

Theoretical analysis given in previous chapter is based on consideration of n line wire. For simplicity, this chapter will deal with numerical calculations of two-conductor system.

The case to which this theory will be applied is induced lightning surges or free oscillations of the lines. When a charged cloud approaches transmission lines, charges appear on the line conductors as bound charges. Now, if the cloud charge is suddenly removed by lightning discharge, the bound charges on the lines can not be fixed and the released bound charges become traveling waves. As an approximate analytical method in such a case, suppose that the initial voltage $E_{t=0}$ is uniformally distributed and there can be no current initially for each line wire.

Then, provided that the conductors are of the different shape and dissymmetrically arranged, as shown in Fig. 2, take the following:

$L_{11}=1.70$ mH/km, $C_{11}=0.00666$ μF/km

$L_{22}=1.52$ mH/km, $C_{22}=0.00746$ μF/km

$L_{12}=L_{21}=0.232$ mH/km,

$C_{12}=C_{21}=-0.00102$ μF/km

$R_{11}=R_{22}=R_{12}=R_{21}=10$ Ω/km

(ground return resistance)

Curves of (24) have been plotted in Fig. 3. for overhead lines 30km long, short-circuited to

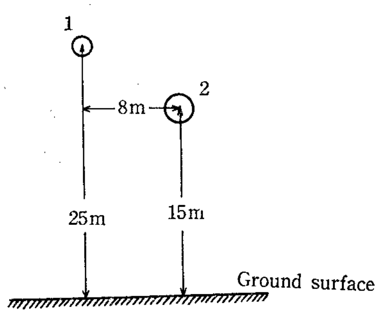
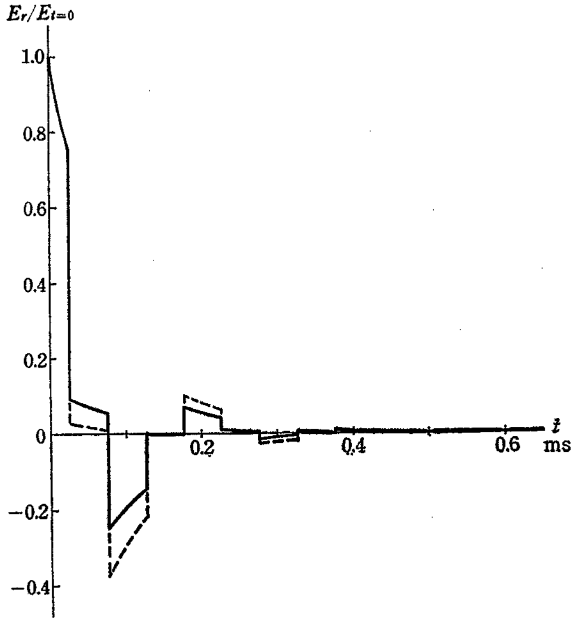
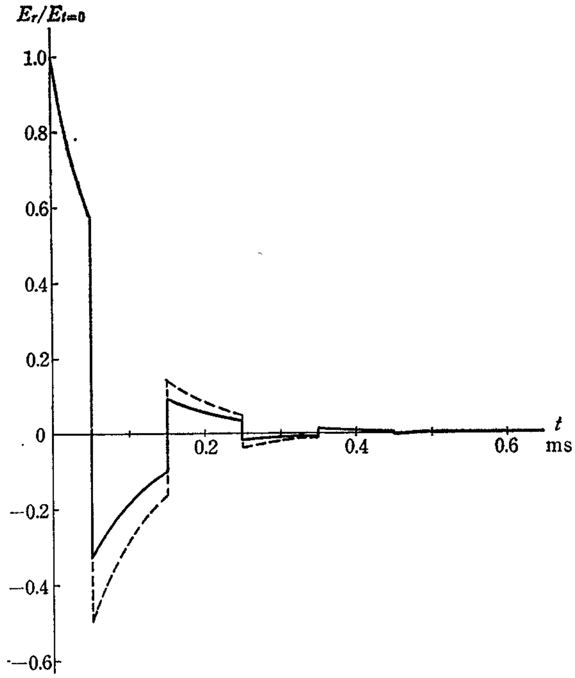


Fig. 2 Group of conductors of radii 1 cm and 1.5 cm.

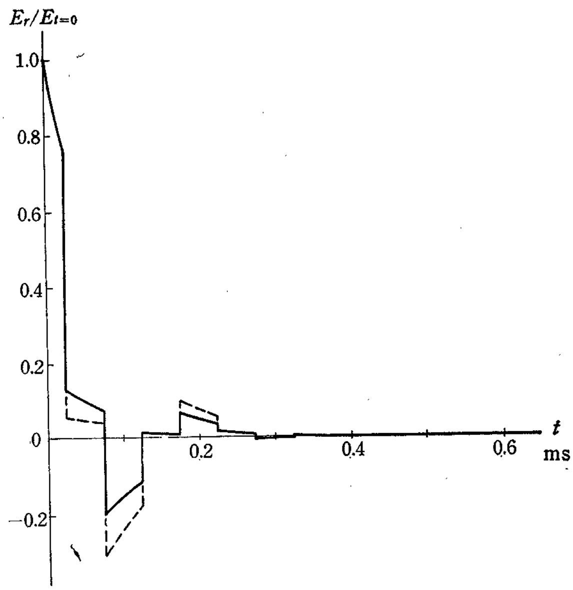


(a) $x=l/4$

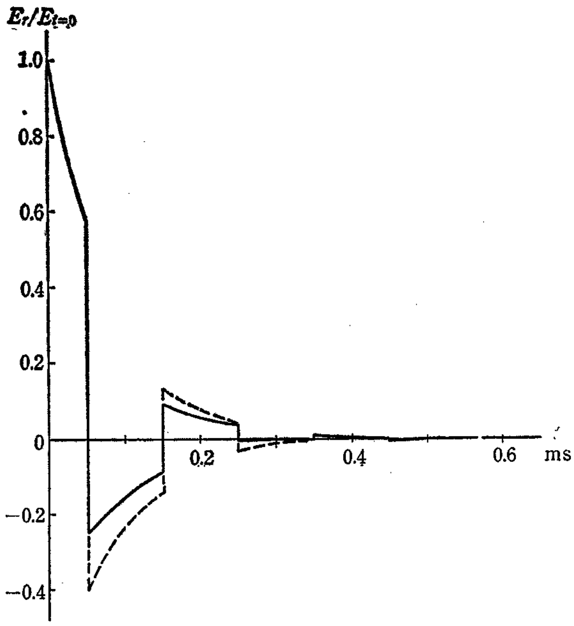


(b) $x=l/2$

Fig. 3 Potentials of the lines, grounded to the earth, as functions of time.
Solid curves: Waves on line wire 1.
Dashed curves: Waves on line wire 2.



(a) $x=l/4$



(b) $x=l/2$

Fig. 4 Potentials of the lines, closed by resistances, as functions of time.
Solid curves: Waves on line wire 1.
Dashed curves: Waves on line wire 2.

the earth. In Fig. 4 are illustrated the potentials of the lines, 30km long, grounded through the resistances at both ends. For this particular example the constants were :

$$R_{0,11}=R_{e,11}=33.5 \quad \Omega$$

$$R_{0,22}=R_{e,22}=30.0 \quad \Omega$$

$$R_{0,12}=R_{0,21}=R_{e,12}=R_{e,21}=0 \quad \Omega$$

There is one aspect of Fig. 3 and Fig. 4 worth pointing out. Each of curves varies with the amplitude of $2l/g$ (g=light velocity). This reason is in the fact that it requires $2l/g$ to go and return for a wave traveling along the line having length l. It should be also remarked that the fast velocity g_1 coincides with the slow velocity g_2, although theoretically the waves moving along the lines have two different velocities and residues will appear in the drived multi-velocity components[1].

V Conclusion

From the foregoing analyses, there appear to be no plausible explanations of the phenomena, although the general solutions of n-conductor system are derived, considering the initial and boundary conditions. By way of illustration of physical concept, the conditions must be simplified, noticing that the conventional traveling wave theory have been based on symmetrical no-loss lines. Physical considerations of Equations (20) for no-loss lines, will be clarified in next report.

References

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