A Trial Production on the Integral II

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(Received November 30, 1962)

In this paper, we define the DC-integral and D_*C-integral by para absolutely continuous functions (ACP functions and ACP_* functions) and study properties of the DC-integral and the D_*C-integral. If a function $f(x)$ is DC (*or* D_*C)-integrable, the value of the DC (*or* D_*C)-integral of $f(x)$ is unique. If a function $f(x)$ is D_*C-integrable on an interval I', then $f(x)$ is DC-integrable on I, and $(DC)\int_I f(x)dx=(D_*C)\int_I f(x)dx$. But there is a function which is DC-integrable and is not D_*C-integrable. If almost everywhere $f(x)=g(x)$ in an interval I, and if $f(x)$ is DC (*or* D_*C)-integrable on I, then $g(x)$ is also DC (*or* D_*C)-integrable on I, and $(DC)\int_I f(x)dx=(DC)\int_I g(x)dx$ (*or* $(D_*C)\int_I f(x)dx=(D_*C)\int_I g(x)dx$). If both $f(x)$ and $g(x)$ are DC-integrable on I, then, for any pair (α, β) of constants, $\alpha f(x)+\beta g(x)$ is DC-integrable on I, and

$$(DC)\int_I(\alpha f(x)+\beta g(x))dx=\alpha\cdot(DC)\int_I f(x)dx+\beta\cdot(DC)\int_I g(x)dx.$$

The same proposition with D_*C in place of DC in the above holds good, too. If $f(x)$ is DC (*or* D_*C)-integrable on an interval I, then $f(x)$ is DC (*or* D_*C)-integrable on the interval $[a, b]$ for any $a, b \in(I-A)$, where A is a enumerable** set. If $f(x)$ is DC-integrable on each of two intervals $[a, c]$ and $[c, b]$, then $f(x)$ is DC-integrable on $[a, b]$ and $(DC)\int_a^b f(x)=(DC)\int_a^c f(x)dx+(DC)\int_c^b f(x)dx$. The same proposition with D_*C in place of DC in the avobe holds good, too. A function $f(x)$ which is $\mathfrak{D}$ (*or* $\mathfrak{D}_*$)-integrable*** on I is DC (*or* D_*C)-integrable on I and we have $(DC)\int_I f(x)dx=(\mathfrak{D})\int_I f(x)dx$ (*or* $(D_*C)\int_I f(x)dx=(\mathfrak{D}_*)\int_I f(x)dx$). If $f(x)$ has the principal value on DC-integral on an interval $[a, b]$, —i.e. if there exists $\lim_{\varepsilon\to 0}\left((DC)\int_a^{c-\varepsilon}f(x)dx+(DC)\int_{c+\varepsilon}^b f(x)dx\right)$ for a point $c(a<c<b)$, then $f(x)$ is DC-integrable on $[a, b]$ and

$$(DC)\int_a^b f(x)dx=\lim_{\varepsilon\to 0}\left((DC)\int_a^{c-\varepsilon}f(x)dx+(DC)\int_{c+\varepsilon}^b f(x)dx.\right.$$

The same proposition with D_*C in place of DC in the above holds good, too.

Throughout this paper, all functions are 1-valued, real valued, and real variable functions, and, open sets and closed sets meen open sets and closed sets in the space consisting of all real numbers respectively unless otherwise specified. The closed interval $\{x|a\leq x\leq b\}$ and the open interval $\{x|a<x<b\}$ are represented by $[a, b]$ and (a, b) respectively.

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** enumeable or finite.

*** $\mathfrak{D}$-integrable = Denjoy integrable in the wide sense.

$\mathfrak{D}_*$-integrable = Denjoy integrable in the restricted sense.

4. DC-integral and D_*C-integral

Theorem 4.1. *A function $F(x)$ which is ACP* on $[a, b]$ almast everywhere approximately derivable** in $[a, b]$.*

[*Proof*] Let A_0 be the non-valued domain of $F(x)$ and $G_i(i=1, 2, 3, \cdots)$ be open sets such that $G_i \subset (a, b)$ & $G_i \supset A_0$ & $mG_i \to 0$ $(i \to \infty)$. Since $F(x)$ is ACP on $[a, b]$, there is a function $F_1(x)$ which is ACG on $[a, b]$ such that $F_1(x)=F(x)$ for every $x \in [a, b] - G_i$. Set $E_i = [a,b] - G_i$, then almost all the points of E_i are points of density for E_i. Since $F(x)=F_1(x)$ for every $x \in E_i$, $F(x)$ is apploximately derivable at almost all $x \in E_i$. Hence $F(x)$ is approximately derivable at almost all $x \in \bigcup_i E_i$. As $m([a, b] - \bigcup_i E_i)=0$, $F(x)$ is approximately derivable at almost all $x \in [a, b]$.

Theorem 4.2. *Let $F(x)$ and $G(x)$ be two functions which are ACP on $[a, b]$, and set $\Phi(x)=F(x)+G(x)$****, then almost everywhere $\Phi'_{ap}(x)=F'_{ap}(x)+G'_{ap}(x)$ in $[a, b]$.*

[*Proof*] Each of $F(x)$ and $G(x)$ is almost everywhere approximately derivable in $[a, b]$. Set $A=\{x \mid F(x)$ is approximately derivable at $x\}$ and $B=\{x \mid G(x)$ is apploximately derivable at $x\}$, then $m([a, b]-(A \cap B))=0$. And $\Phi(x)$ is approximately derivable and $\Phi'_{ap}(x)=F'_{ap}(x)+G'_{ap}(x)$ at each $x \in (A \cap B)$.

Theorem 4.2′. *Let $F(x)$ and $G(x)$ be two functions which are ACP_* on $[a, b]$, and set $\Phi(x)=F(x)+G(x)$, then almost everywhere $\Phi'(x)=F'(x)+G'(x)$ in $[a, b]$.*

Definition 4.1. A function $f(x)$ is termed DC-integrable on an interval I if there is a function $F(x)$ which is ACP on I and which has almost everywhere $F'_{ap}(x)=f(x)$.

Definition 4.1′. A function $f(x)$ is termed D_*C-integrable on an interval I if there is a function $F(x)$ which is ACP_* on I and which has almost everywhere $F'(x)=f(x)$.

Definition 4.2. $F(x)$ in Definition 4.1 is called indefinite DC-integral of $f(x)$ on I.

Definition 4.2′. $F(x)$ in Definition 4.1′ is called indefinite D_*C-integral of $f(x)$ on I.

Theorem 4.3. *Let $f(x)$ be a function which is DC-integrable on $[a, b]$. If both $F(x)$ and $\Phi(x)$ are indefinite DC-integrals of $f(x)$ on $[a, b]$, then we have*

$$F(b)-F(a)=\Phi(b)-\Phi(a).$$

[*Proof*] Set

$$\Psi(x)=F(x)-\Phi(x),$$

then $\Psi(x)$ is ACP on $[a, b]$*****. And, for almost all $x \in [a, b]$,

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* Cf. [2].

** Let $f(x)$ be a measurable function defined on a measurable set E and let x_0 be a point of density for E. If x_0 is a point of density for $\{x \mid l\text{-}\varepsilon \leq F(x)-F(x_0)\ x-x_0 \leq l+\varepsilon,\ x \in E\}$ for a number l and any $\varepsilon>0$, then l is called the approximatly derivatiue of $f(x)$ at x_0 and we denote it by $f'_{ap}(x)=l$.

*** mA is the measur of A.

**** Cf. [2], Definition 2.4.

***** Cf. [2], Theorem 3.5.

$$\Psi'_{\mathrm{ap}}(x)=F'_{\mathrm{ap}}(x)-\Phi'_{\mathrm{ap}}(x)^*.$$

And, from the assumption,

$$F'_{\mathrm{ap}}(x)=f(x) \quad \text{a.e. in } [a, b],$$

$$\Phi'_{\mathrm{ap}}(x)=f(x) \quad \text{a.e. in } [a, b].$$

Hence we have

$$\Psi'_{\mathrm{ap}}(x)=0 \quad \text{a.e. in } [a, b].$$

Hence $\Psi(x)$ is a constant.** Hence we have

$$\Psi(b)-\Psi(a)=0$$

Hence

$$(F(b)-\Phi(b))-(F(a)-\Phi(a))=0.$$

Hence we have

$$F(b)-F(a)=\Phi(b)-\Phi(a). \qquad Q.E.D.$$

By a similar argument, we have the following theorem.

Theorem 4.3′. *Let $f(x)$ be a function which is* $\mathrm{D}_*\mathrm{C}$*-integrable on $[a, b]$. If both $F(x)$ and $\Phi(x)$ are indefinite* $\mathrm{D}_*\mathrm{C}$*-integrals of $f(x)$, then*

$$F(b)-F(a)=\Phi(b)-\Phi(a).$$

Difinition 4.3. Let $f(x)$ be a function which is DC-integrable on $I=[a, b]$ and $F(x)$ be a indefinite DC-integral of $f(x)$ on I. Then, $F(b)-F(a)$ is termed definite DC-integral of $f(x)$ over I and is denoted by

$$(\mathrm{DC})\int_I f(x)dx \quad \text{or} \quad (\mathrm{DC})\int_a^b f(x)dx.$$

Definition 4.3′. Let $f(x)$ be a function which is $\mathrm{D}_*\mathrm{C}$-integrable on $I=[a, b]$ and $F(x)$ be a indefinite $\mathrm{D}_*\mathrm{C}$-integrals of $f(x)$ on I. Then, $F(b)-F(a)$ is termed definite $\mathrm{D}_*\mathrm{C}$-integral of $f(x)$ over I and is denoted by

$$(\mathrm{D}_*\mathrm{C})\int_I f(x)dx \quad \text{or} \quad (\mathrm{D}_*\mathrm{C})\int_a^b f(x)dx.$$

Theorem 4.4. *If $f(x)$ is* $\mathrm{D}_*\mathrm{C}$*-integrable on $[a, b]$, then $f(x)$ is* DC*-integrable on $[a, b]$ and*

$$(\mathrm{DC})\int_a^b f(x)=(\mathrm{D}_*\mathrm{C})\int_a^b f(x)dx.$$

[*Proof*] From definitions.

Theorem 4.5. *Let $f(x)$ and $g(x)$ be two functions each of which is defined almost everywhere in an interval I. If almost everywhere $f(x)=g(x)$ in I and if $f(x)$ is*

* Cf. Theorem 4.1

** Cf. [2] Theorem 3.12.

DC-*integrable on I, then* $g(x)$ *is also* DC-*integrable on I and*

$$(\mathrm{DC})\int_I f(x)dx=(\mathrm{DC})\int_I g(x)dx.$$

The same proposition with D$_*$C *in place of* DC *holds good, too.*

Theorem 4.6. *If two functions* $f(x)$ *and* $g(x)$ *are* DC-*integrable on an interval I, then any linear combination* $af(x)+bg(x)$ *of these functions is* DC-*integrable on I and we have*

$$(\mathrm{DC})\int_I (af(x)+bg(x))dx=a(\mathrm{DC})\int_I f(x)dx+b(\mathrm{DC})\int_I g(x)dx.$$

The same proposition with D$_*$C *in place of* DC *holds good, too.*

[*Proof*] Let $F(x)$ and $G(x)$ be indefinite DC-integrals of $f(x)$ and $g(x)$ on I respectively. Then, $\Phi(x)=aF(x)+bG(x)$ is ACP on I and we have

$$\Phi'_{\mathrm{ap}}(x)=aF'_{\mathrm{ap}}(x)+bG'_{\mathrm{ap}}(x) \quad \text{a. e.} \quad \text{in } I.$$

And,

$$aF'_{\mathrm{ap}}(x)+bG'_{\mathrm{ap}}(x)=af(x)+bg(x) \quad \text{a.e. in } I.$$

Hence

$$(\mathrm{DC})\int_I (af(x)+bg(x))dx=a\cdot(\mathrm{DC})\int_I f(x)dx+b\cdot(\mathrm{DC})\int_I g(x)dx.$$

By a similar argument, we have a proof of the same proposition with D$_*$C in place of DC.

Theorem 4.7. *If* $f(x)$ *is* DC-*integrable on* $I=[a, b]$, *then* $f(x)$ *is* DC-*integrable on* $[c, d]$ *for any pair* (c, d) $(c, d\in(I-A))$ *where A is an enumerable set.*

The same proposition with D$_*$C *in place of* DC *holds good, too.*

[*Proof*] The non-valued domain* of indefinite DC-integral of $f(x)$ on $[a, b]$ is an enumerable (or finite) set.

Theorem 4.8. *If* $f(x)$ *is* DC-*integrable on each of* $[a, b]$ *and* $[b, c]$, *then* $f(x)$ *is* DC-*integrable on* $[a, c]$ *and*

$$(\mathrm{DC})\int_a^c f(x)dx=(\mathrm{DC})\int_a^b f(x)dx+(\mathrm{DC})\int_b^c f(x)dx.$$

The same proposition with D$_*$C *in place of* DC *holds good, too.*

Theorem 4.9. *If both* $f(x)$ *and* $g(x)$ *are* DC-*integrable on* $[a, b]$ *and if* $f(x)\geq g(x)$ *almost everywhere in* $[a, b]$, *then*

$$(\mathrm{DC})\int_a^b f(x)dx\geq(\mathrm{DC})\int_a^b g(x)dx.$$

The same proposition with D$_*$C *in place of* DC *holds good, too.*

[*Proof*] Since $f(x)$ is DC-integrable on $[a, b]$, there is a function $F(x)$ which is ACP on $[a, b]$ such that $F'_{\mathrm{ap}}(x)=f(x)$ almost everywhere in $[a, b]$. And, since $g(x)$ is DC-

* Cf. [2], Definition 2.3.

integrable on $[a, b]$ there is a function $G(x)$ which is ACP on $[a, b]$ such that $G'_{ap}(x) = g(x)$ almost everywhere in $[a, b]$. Set $\Phi(x)=F(x)-G(x)$ and $\varphi(x)=f(x)-g(x)$, then $\Phi(x)$ is ACP on $[a, b]$ and $\Phi'_{ap}(x)=\varphi(x)$ almost everywhere in $[a, b]$. $\varphi(x)$ is non-negative almost everywhere in $[a, b]$. Hence, by Theorem 3.11 in [2], $\Phi(x)$ is monotone non-decreasing. Hence

$$0 \leq \Phi(b)-\Phi(a)=(\mathrm{DC})\int_a^b \varphi(x)dx=(\mathrm{DC})\int_a^b f(x)dx-(\mathrm{DC})\int_a^b g(x)dx.$$

Hence we have

$$(\mathrm{DC})\int_a^b f(x)dx \geq (\mathrm{DC})\int_a^b g(x)dx.$$

By a similar argument, we have a proof of the same proposition with $\mathrm{D_*C}$ in place of DC.

Theorem 4.10. *If a function (x) defined almost everywhere in $[a, b]$ has $f(x)=0$ almost everywhere in $[a, b]$, $f(x)$ is* $\mathrm{D_*C}$*-integrable—and hence* DC*-integrable—and*

$$(\mathrm{D_*C})\int_a^b f(x)dx=(\mathrm{DC})\int_a^b f(x)dx=0.$$

[*Proof*] A function $F(x)$ having $F(x)=0$ for all $x \in [a, b]$ is a indefinite $\mathrm{D_*C}$-integral of $f(x)$. And $F(b)-F(a)=0$.

Theorem 4.11. *Let $f(x)$ be a function defined almost everywhere in $[a, b]$.*

i) *If A is a set which is dence in an open interval (a, a') $(a<a'<b)$ and if $f(x)$ is* DC*-integrable on each $[s, b]$ $(S \in A)$ and if there is*

$$\lim_{(s;s\in A)\to a}^{*} (\mathrm{DC})\int_s^b f(x)dx \quad (\neq \pm\infty),$$

then $f(x)$ is DC*-integrable on $[a, b]$ and*

$$(\mathrm{DC})\int_a^b f(x)dx = \lim_{(s \mid s\in A)\to a} (\mathrm{DC})\int_s^b f(x)dx.$$

ii) *If B is a set which is dence in an open interval (b', b) $(a<b'<b)$ and if $f(x)$ is* DC*-integrable on each $[a, t]$ $(t \in B)$ and if there is*

$$\lim_{(t \mid t\in B)\to b} (\mathrm{DC})\int_a^t f(x) \quad (\neq \pm\infty),$$

then $f(x)$ is DC*-integrable on $[a, b]$ and*

$$(\mathrm{DC})\int_a^b f(x)dx = \lim_{(t \mid t\in B)\to b} \int_a^t f(x)dx.$$

iii) *If A and B are sets which are dence in open intervals (a, a') and (b', b) $(a<a'<b'<b)$ respectively and if $f(x)$ is* DC*-integrable on each $[s, t]$ $(s \in A,\ t \in B)$ and if there is*

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* Cf. [2], Definition 2.1.

$$\lim_{\substack{(s \mid s \in A) \to a \\ (t \mid t \in B) \to b}} (\mathrm{DC})\int_s^t f(x)dx \quad (\neq \pm\infty),$$

then $f(x)$ *is* DC-*integrable on* $[a, b]$ *and*

$$(\mathrm{DC})\int_a^b f(x)dx = \lim_{\substack{(s \mid s \in A) \to a \\ (t \mid t \in B) \to b}} (\mathrm{DC})\int_s^t f(x)dx.$$

The same proposition with $\mathrm{D_*C}$ *in place of* DC *holds good, too.*

[*Proof*] i) Let $\{a_n\}$ be a sequence such that $a_1 > a_2 > \cdots \to a$ & $a_n \in A (n=1, 2, \cdots)$. Since $f(x)$ is DC-integrable on $[a_n, b] (n=1, 2, \cdots)$, there is a function $F_n(x)$ which is ACP on $[a_n, b]$ such that

$$(F_n)'_{\mathrm{ap}}(x) = f(x) \quad \text{a.e. in } [a_n, b].$$

We may assume that $F_n(a_{n-1}) = F_{n-1}(a_{n-1})$. As there is $\lim_{n\to\infty}(\mathrm{DC})\int_{a_n}^b f(x)dx$, hence there is $\lim_{n\to\infty}(F_n(b) - F_n(a_n))$. As $F_1(b) = F_2(b) = \cdots$, hence there is $\lim_{n\to\infty} F_n(a_n)$.

Set

$F(a) = \lim_{n\to\infty} F_n(a_n)$,

$F(x) = F_1(x)$ for $x \in$ (the domain of definition of $F_1(x)$),

$F(x) = F_n(x)$ for $x \in$ ((the domain of difinition of $F_n(x)$)

$-$(the domain of definition of $F_{n-1}(x)$)) $(n=2, 3, \cdots)$.

We shall paove that $F(x)$ is ACP on $[a, b]$.

The non-valued domain D_0 of $F(x)$ is equal to $\bigcup_n$ (the non-valued domain of $F_n(x)$), hence D_0 is a scattered set.

We shall prove that $F(x)$ is continuous on the domain $D = [a, b] - D_0$ of the definition. It is obvious that $F(x)$ is continuous at each $x (x \in D$ & $x \neq a)$. We shall prove that $F(x)$ is continuous at $x = a$. By the assumption, $\lim_{(x \mid x \in A) \to a} F(x) = F(a)$. Hence, for any $\varepsilon > 0$, there is a number $\delta > 0$ such that $|F(x) - F(a)| < \frac{\varepsilon}{2}$ for all $x (x - a < \delta$ & $x \in A \cap D)$. We may assume $\delta < a' - a$. For any $p \in D - A (0 < p - a < \delta)$, there is an integer n such that $p \in (a_n, b)$. Since $F_n(x) = F(x)$ for every $x \in$ (the domain D_n of definition of $F_n(x)$), we have $|F(x) - F(a)| < \frac{\varepsilon}{2}$ for all $x (x - a < \delta$ & $x \in A \cap D_n)$. Since $F_n(x)$ is continuous on D_n, we have $|F_n(p) - F(a)| \leq \frac{\varepsilon}{2} < \varepsilon$.

Hence $F(x)$ is contiunous at a. Hence $F(x)$ is continuous on D. From the fact that $F(x)$ is continuous on D and the definition of $F(x)$, we can easily conclued that $F(x)$ is para contiunous on $[a, b]$.

Secondly, we shall prove that $F(x)$ is ACP on $[a, b]$.

Let G be an open set which contains the non-valued domain of $F(x)$. Then. $G' = G - ((\bigcup_n a_n) \cup a \cup b)$ is also an open set which containts the non-valued domain of $F(x)$. And, $G' \cap [a_n, b]$ is an open set which contatins the non-valued domain of $F_n(x)$. Since $F_n(x)$ is ACP $[a_n, b]$, there is a function $\Phi_n(x)$ which is ACG on $[a_n, b]$ such that $\Phi_n(x) = F_n(x)$ for each $x \in ([a_n, b] - G')$. Represent the open set $G' \cap [a_n, b]$ by the union $\bigcup_{i=1}^{\infty} (c_i, c_i') ((c_i, c_i') \cap (c_j, c_j') = 0$ if $i \neq j)$ of an enumerable of open intervals, and set

$$\Psi_n(x)=\Phi_n(x) \quad \text{for} \quad x\in([a_n, b]-\bigcup_i(c_i, c_i')),$$
$$\Psi_n(x)=\Phi_n(c_i)+\frac{\Phi_n(c_i')-\Phi(c_i)}{c_i'-c_i}(x-c_i) \quad \text{for} \quad x\in(c_i, c_i'),$$
$$(i=1, 2, \cdots),$$

then $\Psi_n(x)$ is ACG on $[a_n, b]$. And, $\Psi_m(x)=\Psi_n(x)$ for each $x\in[\mathrm{Max}(a_m, a_n), b]$. Set

$$\Psi(a)=F(a) \qquad (=\lim_{n\to\infty} F_n(a_n)),$$
$$\Psi(x)=\Phi_n(x) \quad \text{for} \quad x\in[a_n, b].$$

Then, we can prove that $\Psi(x)$ is continuous on $[a, b]$ by a fact that $F(x)$ is continuous on D. Hence $\Psi(x)$ is ACG on $[a, b]$. And, $\Psi(x)=F(x)$ for each $x\in([a, b]-G')$. As $G'\subset G$, $\Psi(x)=F(x)$ for each $x\in([a, b]-G)$. Hence $F(x)$ is ACP on $[a, b]$. Hence $f(x)$ is DC-integrable on $[a, b]$ and

$$(\mathrm{DC})\int_a^b f(x)dx=F(b)-F(a)=F(b)-\lim_{n\to\infty} F_n(a_n)=\lim_{n\to\infty}(F_n(b)-F_n(a_n))$$
$$=\lim_{n\to\infty}(\mathrm{DC})\int_{a_n}^b f(x)dx=\lim_{(s\mid s\in A)\to a}(\mathrm{DC})\int_s^b f(x)dx.$$

We can prove ii) and iii) by similar arguments.

By similer arguments, we have proof of the same proposition with $\mathrm{D}_*\mathrm{C}$ in place of DC.

Theorem 4.12. *A function $f(x)$ which is* DC-*integrable on $[a, b]$ is necessarily measurable and almost everywhere finite in $[a, b]$.*

[*Proof*] Since $f(x)$ is DC-integrable, there is a function $F(x)$ which is ACP on $[a, b]$ such that almost everywhere $F'_{\mathrm{ap}}(x)=f(x)$ in $[a, b]$. Let A be the non-valued domain and let $\{G_n|n=1, 2, \cdots\}$ be a sequence of open sets such that $G_n\supset A(n=1, 2, \cdots)$ & $G_n\subset(a, b)$ $(n=1, 2, \cdots)$ & $mG_n\to 0$. For each G_n, there is a function $F_n(x)$ which is ACG on $[a, b]$ such that $F_n(x)=F(x)$ for each $x\in[a, b]-G_n$. Set $H_n=[a, b]-G_n$; then almost everywhere $(F_n)'_{\mathrm{ap}}(x)=F'_{\mathrm{ap}}(x)=f(x)$ in H_n^*. Hence $f_{(H_n)}(x)$** is measurable and almost everywhere finite in H_n. Hence $f_{(\bigcup_n H_n)}(x)$ is measurable and almost everywhere finite in $\bigcup_n H_n$. As $m([a, b]-\bigcup_n H_n)=0$, $f(x)$ is measurable and almost everywhere finite in $[a, b]$.

5. Relations with Denjoy integral and principal value of integral

Theorem 5.1. *If a function $f(x)$ is $\mathfrak{D}$-integrable on $[a, b]$, then $f(x)$ is* DC-*integrable on $[a, b]$ and*

$$(\mathrm{DC})\int_a^b f(x)dx=(\mathfrak{D})\int_a^b f(x)dx.$$

[*Proof*] Any function which is ACG on $[a, b]$ is ACP on $[a, b]$.

* Cf. The proof of Theorem 4.1.

** Let $\varphi(x)$ be a function defined on a set X and let $X'\subset X$. We denote the following function $\psi(x)$ defined on X' by $\varphi_{(X')}(x)$: $\psi(x)=\varphi(x)$ for each $x\in X'$.

Theorem 5.1′. *If a function $f(x)$ is $\mathfrak{D}_*$-integrable on $[a, b]$, then $f(x)$ is* $\mathrm{D_*C}$*-inetgrable on $[a, b]$ and*

$$(\mathrm{D_*C})\int_a^b f(x)dx=(\mathfrak{D}_*)\int_a^b f(x)dx.$$

Theorem 5.2. *A function $f(x)$ which is* DC*-integrable and is non-negative almost everywhere in $I=[a, b]$ is necessarily $\mathfrak{L}$-integrable* on I.*

[*Proof*] Let $F(x)$ be an indefinite DC-integral, then $F(x)$ is approximately derivable almost everywhere in I and

$$F'_{\mathrm{ap}}(x)=f(x) \quad \text{a.e. in } I.$$

Since $f(x)\geq 0$ a.e. in I, by Theorem 3.9 in [2], $F(x)$ can be extended to a function $\Phi(x)$ which is ACG on $[a, b]$. Since

$$\Phi'_{\mathrm{ap}}(x)=F'_{\mathrm{ap}}(x)=f(x) \quad \text{a.e. in } I,$$

we have

$$(\mathfrak{D})\int_I f(x)dx=\Phi(b)-\Phi(a)=F(b)-F(a)=(\mathrm{DC})\int_I f(x)dx.$$

Theorem 5.3. *There is a function which is* DC (or $\mathrm{D_*C}$)*-integrable and in not $\mathfrak{D}$* (or $\mathfrak{D}_*$)*-integrable on $[a, b]$.*

[*Proof*] There are many examples.

(**Remark**) Example 1. $f(0)=0$,

$$f(x)=\frac{1}{x} \quad \text{for} \quad 0<|x|\leq 1.$$

$$(\mathrm{D_*C})\int_{-1}^1 f(x)dx=(\mathrm{DC})\int_{-1}^1 f(x)dx=0.$$

$f(x)$ is not $\mathfrak{D}$-integrable on $[-1, 1]$.

Example 2.

$$f(x)=\frac{1}{2^{2n}x+2} \quad \text{for} \quad \frac{-3}{2^{2n}}\leq x\leq\frac{-1}{2^{2n}} \;\&\; x\neq\frac{-2}{2^{2n}},$$

$$f(x)=0 \text{ for } \frac{-1}{2^{2n}}<x<\frac{-3}{2^{2(n+1)}} \text{ or } x=\frac{-2}{2^{2n}} \text{ or } \frac{3}{2^{2(n+1)}}<x<\frac{1}{2^{2n}}$$

$$\text{or } x=\frac{2}{2^{2n}} \text{ or } x=0,$$

$$f(x)=\frac{1}{2^{2n}x-2} \text{ for } \frac{1}{2^{2n}}\leq x\leq\frac{3}{2^{2n}} \;\&\; x\neq\frac{2}{2^{2n}},$$

$$(n=1, 2, 3, \cdots).$$

$$(\mathrm{D_*C})\int_{-3/4}^{3/4} f(x)dx=(\mathrm{DC})\int_{-3/4}^{3/4} f(x)dx=0.$$

$f(x)$ is not $\mathfrak{D}$-integrable and also has not the principal value* on $\mathfrak{D}$-integral.

Theorem 5.4. *If a function $f(x)$ has the principal value on* DC*-integral on $[a, b]$,*

* $\mathfrak{L}$-integrable = Lebesgue integrable.

** $\lim_{\varepsilon\to 0}\left((\mathfrak{D})\int_a^{c-\varepsilon} f(x)dx+(\mathfrak{D})\int_{c+\varepsilon}^b f(x)dx\right).$

i.e. if there is

$$(\mathrm{P})\underset{c}{\int_a^b} f(x)dx=\lim_{\varepsilon\to 0}((\mathrm{DC})\int_a^{c-\varepsilon} f(x)+(\mathrm{DC})\int_{c+\varepsilon}^b f(x)dx)^*$$

for a number $c(a<c<b)$, then $f(x)$ is DC-*integrable on* $[a, b]$ *and*

$$(\mathrm{DC})\int_a^b f(x)dx=(\mathrm{P})\underset{c}{\int_a^b} f(x)dx.$$

The same proposition with $\mathrm{D}_*\mathrm{C}$ *in place of* DC *holds good, too.*

[*Proof*] Set

$F(x)=(\mathrm{DC})\int_a^x f(t)dt$ for each $x(a\leq x<c)$ & $f(t)$ is DC-integrable on $[x, c]$),

$F(x)=(\mathrm{P})\underset{c}{\int_a^b} f(x)dx-(\mathrm{DC})\int_x^b f(t)dt$ for each $x(c<x\leq b)$ & $f(t)$ is DC-integrable on $[x, b]$).

Tnen, $F(x)$ is continuous on $(a,b)-A$ where A is a scattered set.** And

$$\lim_{(\varepsilon|\, c+\varepsilon,\ c-\varepsilon\in(\text{the domain of definition of } F(x))\to 0} (F(c+\varepsilon)-F(c-\varepsilon))$$
$$=\lim_{\varepsilon\to 0}((\mathrm{P})\underset{c}{\int_a^b} f(t)dt-(\mathrm{DC})\int_{c+\varepsilon}^b f(t)dt-(\mathrm{DC})\int_a^{c-\varepsilon} f(t)dt=0.$$

Hence $F(x)$ is para continuous on $[a, b]$. We can easily prove that $F(x)$ is ACP on $[a, b]$. And

$$F'_{\mathrm{ap}}(x)=f(x) \quad \text{a.e. in } [a, b].$$

Hence $f(x)$ is DC-integrable on $[a, b]$ and

$$(\mathrm{DC})\int_a^b f(x)dx=F(b)-F(a)=(\mathrm{P})\underset{c}{\int_a^b} f(x)dx-(\mathrm{DC})\int_b^b f(t)dt-(\mathrm{DC})\int_a^a f(t)dt$$
$$=(\mathrm{P})\underset{c}{\int_a^b} f(x)dx.$$

We have a proof of the same proposition on $\mathrm{D}_*\mathrm{C}$ by a similar argument.

Theorem 5.5. *Given a non-decreasing sequence $\{f_n(x)\}$ of functions which are* DC-*integrable on an interval I and whose definite* DC-*integrals over I constitute a sequence bounded above, the function $f(x)=\lim_{n\to\infty} f_n(x)$ is* DC-*integrable on I and*

$$(\mathrm{DC})\int_I f(x)dx=\lim_n (\mathrm{DC})\int_I f_n(x)dx.$$

The same proposition with $\mathrm{D}_*\mathrm{C}$ *in place of* DC *holds good, too.*

[*Proof*] $f_n(x)-f_1(x)$ $(n=2, 3, \cdots)$

is DC-integrable on I, and $f_n(x)-f_1(x)\geq 0$. Hence, by Theorem 5.2, $f_n(x)-f_1(x)$ is $\mathfrak{D}$-integrable on I. Since DC-integrals over I of $\{f_n(x)\,|\,n=1, 2, \cdots\}$ constitute a sequence bounded above, there is a number G such that

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* Now we write (ε| There are both $(\mathrm{DC})\int_a^{c-\varepsilon} fdx$ and $(\mathrm{DC})\int_{c+\varepsilon}^b fdx)\to 0$ by $\varepsilon\to 0$.

** $A\ni c$.

$$G > (\mathrm{DC})\int_I f_n(x)dx \quad (n=1, 2, \cdots).$$

Hence

$$(\mathfrak{D})\int_I (f_n(x)-f_1(x))dx = (\mathrm{DC})\int_I (f_n(x)-f_1(x))dx$$
$$=(\mathrm{DC})\int_a^b f_n(x)dx-(\mathrm{DC})\int_a^b f_1(x)dx < G-(\mathrm{DC})\int_a^b f_1(x)dx.$$

Hence

$\left\{(\mathfrak{D})\int_a^b (f_n(x)-f_1(x))dx \,|\, n=1, 2, \cdots\right\}$ is bounded above, hence

$$(\mathfrak{D})\int_a^b \lim_{n\to\infty}(f_n(x)-f_1(x))dx = \lim_{n\to\infty}(\mathfrak{D})\int_a^b (f_n(x)-f_1(x))dx.$$

Hence

$$(\mathrm{DC})\int_a^b (f(x)-f_1(x))dx = (\mathrm{DC})\int_a^b \lim_{n\to\infty} (f_n(x)-f_1(x))dx$$
$$=\lim_{n\to\infty}(\mathrm{DC})\int_a^b (f_n(x)-f_1(x))dx = \lim_{n\to\infty}(\mathrm{DC})\int_a^b f_n(x)dx-(\mathrm{DC})\int_a^b f_1(x)dx.$$

Hence we have

$$(\mathrm{DC})\int_a^b f(x)dx = \lim_{n\to\infty}(\mathrm{DC})\int_a^b f_n(x)dx.$$

We have a proof of the same proposition on $\mathrm{D_*C}$ by a similar argument.

Theorem 5.6. *If a function $f(x)$ is DC-integrable on $[a, b]$, then, for any pair (a', b') $(a \leq a' < b' \leq b)$, there are a'' and b'' $(a' \leq a'' < b'' \leq b')$ such that $f(x)$ is $\mathfrak{L}$-integrable on $[a'', b'']$.*

[*Proof*] Let $F(x)$ be an indefinite DC-integral on $[a, b]$. Since the non-valued domain A of $F(x)$ is scattered, $\overline{A}$ is nowhere dence. Hence, for a' and b', there are a_1' and b_1' $(a' \leq a_1' < b_1' \leq b')$ such that $F(x)$ is ACG on $[a_1', b_1']$. Hence $f(x)$ is $\mathfrak{D}$-integrable on $[a_1', b_1']$. Hence there are a'' and b'' $(a_1' \leq a'' < b'' \leq b_1')$ such that $f(x)$ is $\mathfrak{L}$-integrable on $[a'', b'']$.

The integration by parts, Stieltjes integral, and constructive definition of DC-integral shall be studied in Part III.

References

1) S. Saks: Theory of the integral, Warszawa (1937).
2) Y. Hayashi: A trial production on the integral I. Bull. Univ. Osaka Pref. Ser. **A. vol. 11** N.o 1 (1962) 121-131.

Corrections to "A taial Production on the Integral I". Page 130, line 27, "Hence" should read "Because". Page 129, line 3, "null and" should be eliminate.