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The tension problem of the perforated strip, which is stiffened with flanges along both edges, is studied. The calculation is executed for typical three cases and results of numerical calculation for box-shaped beam are shown.

1. Introduction

The stress analysis of a perforated strip under tension has been studied by many investigators, with fruitful results. But the effects of the flanges, when they are attached to the straight edges of the perforated strip, on the stress distribution are left unclarified, while this problem is one of theoretical interest as well as practical importance.

In the present paper, the tension problem of the perforated strip, which has one circular hole mid-way between the edges and is stiffened symmetrically with the flanges along both edges, is studied. It is dealt as a two-dimensional problem and its solution is sought by successive method for typical three cases. This process is analogous to that used by Howland[1] for the unflanged strip.

2. Theory

We consider the perforated strip with the flanges on both edges, of isotropic, elastic material, infinitely extended in longitudinal direction and loaded by uniform tension

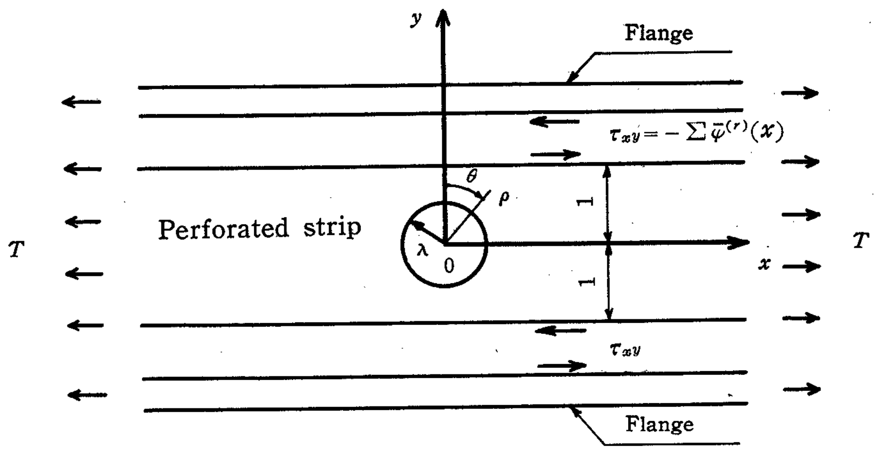

Fig. 1. Perforated strip

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applied at infinity.

As shown in Fig. 1, Cartesian co-ordinates (x, y) has its origin at the center of the circular hole (its radius is equal to λ) and the x-axis lies on the center line of the perforated strip. As the stress system is symmetrical both about the axes of the co-ordinates, the analysis will be developed only for the positive region of x and y. Co-ordinates x, y and the radius λ are measured in a unit equal to half the width of the perforated strip.

Both the perforated strip and the flanges are regarded as thin plates (the thicknesses of which are equal to t_w and t_f, respectively) and the normal stresses perpendicular to the plane of the plate are ignored.

Separating the flanges from the perforated strip, each of them is supposed to be in a state of generalised plane stress. The flanges and the perforated strip are subjected to shearing stresses at the lines of connection between them and these shearing stresses are determined from the condition of continuity of normal strains at their intersection.

2-1 Stress Function of the Perforated Strip (Successive Approximation)

Polar co-ordinates (ρ, θ) will also be used, and it will be convenient to take the initial line along the y-axis and the positive direction of θ clockwise. Then the relation between the two systems of co-ordinates is

$$x=\rho\sin\theta, \quad y=\rho\cos\theta \tag{1}$$

where ρ is co-ordinate measured in a unit equal to half the width of the perforated strip.

Denoting the stress function, χ, χ must satisfy the following conditions (a), (b), (c) and (d):

(a) At all points within the material

$$\frac{\partial^4\chi}{\partial x^4}+2\frac{\partial^4\chi}{\partial x^2\partial y^2}+\frac{\partial^4\chi}{\partial y^4}=0; \tag{2}$$

(b) When $x\to\infty$, the stresses are

$$\sigma_x=\frac{\partial^2\chi}{\partial y^2}=T,\ \sigma_y=\frac{\partial^2\chi}{\partial x^2}=0,\ \tau_{xy}=-\frac{\partial^2\chi}{\partial x\partial y}=0; \tag{3}$$

(c) On the straight edge, $y=1$,

$\sigma_y=0$ and the normal strain in the x-direction must be equal to that occuring in the flange at the line of connection between the flange and the perforated strip. (4)

(d) On the edge of the hole, $\rho=\lambda$,

$$\left.\begin{aligned}\sigma_\rho&=\frac{1}{\rho^2}\frac{\partial^2\chi}{\partial\theta^2}+\frac{1}{\rho}\frac{\partial\chi}{\partial\rho}=0\\ \tau_{\rho\theta}&=-\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\chi}{\partial\theta}\right)=0\end{aligned}\right\}. \tag{5}$$

To satisfy these conditions, we write

$$\chi=\chi_0'+\chi_0+\chi_1+\chi_2+\cdots\cdots, \tag{6}$$

where the terms of the series are each, separately, solution of the biharmonic equation (2) and have, in addition, the following properties.

χ_0' gives the stresses at infinity.

$\chi_0'+\chi_0$ satisfies the conditions on the edge of the hole and at infinity, but not on the straight edge, i.e., it is the solution for an infinite plate with a hole.

χ_1 cancels the stresses due to χ_0 on the edge $y=1$ and satisfies the condition of the continuity of the normal strain to the flange on the edge $y=1$, but introduces stresses on the edge of the hole.

χ_2 cancels these, but again does not satisfy the boundary conditions on the straight edge.

More generally, $\chi_{2r}+\chi_{2r+1}$ satisfies the boundary conditions on $y=1$, while $\chi_{2r-1}+\chi_{2r}$ gives zero stresses over $\rho=\lambda$.

Now we derive equations from which χ_{2r+1} and χ_{2r+2} may be calculated. The value of χ_{2r} will be assumed to be given in the form

$$\chi_{2r}=-D_0^{(r)}\log\rho+\sum_{n=1}^{\infty}\left(\frac{D_{2n}^{(r)}}{\rho^{2n}}+\frac{E_{2n}^{(r)}}{\rho^{2n-2}}\right)\cos 2n\theta, \tag{7}$$

where $D_0^{(r)}$, $D_{2n}^{(r)}$, $E_{2n}^{(r)}$ are coefficients, to be determined later.

For simplicity of writing, it will be convenient to omit the suffix (r) in all the coefficients until it becomes necessary to distinguish them from those of the χ_{2r+2} series. Making this temporary simplification of notation, we have for the stresses due to χ_{2r},

$$\sigma_\rho=\frac{1}{\rho}\frac{\partial\chi_{2r}}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2\chi_{2r}}{\partial\theta^2}$$
$$=-\frac{D_0}{\rho^2}-2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}}{\rho^{2n+2}}+\frac{(n+1)(2n-1)E_{2n}}{\rho^{2n}}\right\}\cos 2n\theta \tag{8}$$

$$\tau_{\rho\theta}=-\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\chi_{2r}}{\partial\theta}\right)$$
$$=-2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}}{\rho^{2n+2}}+\frac{n(2n-1)E_{2n}}{\rho^{2n}}\right\}\sin 2n\theta, \tag{9}$$

$$\sigma_\theta=\frac{\partial^2\chi_{2r}}{\partial\rho^2}$$
$$=\frac{D_0}{\rho^2}+2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}}{\rho^{2n+2}}+\frac{(n-1)(2n-1)E_{2n}}{\rho^{2n}}\right\}\cos 2n\theta. \tag{10}$$

The stresses relative to the Cartesian axes are given by the equations

$$\sigma_x=\frac{D_0-2E_2}{\rho^2}\cos 2\theta+2\sum_{n=1}^{\infty}\left\{\frac{(2n+1)(nD_{2n}-E_{2n+2})}{\rho^{2n+2}}+\frac{n(2n-1)E_{2n}}{\rho^{2n}}\right\}\cos(2n+2)\theta \tag{11}$$

$$\sigma_y=-\frac{D_0+2E_2}{\rho^2}\cos 2\theta-2\sum_{n=1}^{\infty}\left\{\frac{(2n+1)(nD_{2n}+E_{2n+2})}{\rho^{2n+2}}+\frac{n(2n-1)E_{2n}}{\rho^{2n}}\right\}\cos(2n+2)\theta, \tag{12}$$

$$\tau_{xy}=-\left[\frac{D_0}{\rho^2}\sin 2\theta+2\sum_{n=1}^{\infty}\left\{\frac{n(2n+1)D_{2n}}{\rho^{2n+2}}+\frac{n(2n-1)E_{2n}}{\rho^{2n}}\right\}\sin(2n+2)\theta\right]. \tag{13}$$

We have now to construct χ_{2r+1} so that it produces on the edge of the strip, the stresses cancelling the value of σ_y in (12), and making the value of shearing stress to be $-\overline{\psi}(x)$ after addition of the value of τ_{xy} in (13), where $-\overline{\psi}(x)$ is to be determined from the condition of the continuity of the normal strain; i.e., χ_{2r+1} satisfies

$$\left.\begin{aligned} &\sigma_y=\phi(x),\ \tau_{xy}=\psi(x)-\overline{\psi}(x) \quad \text{on} \quad y=1, \\ \text{where} \qquad &\phi(x)=-[\sigma_y \text{ in } (12)]_{y=1}\text{: even function}, \\ &\psi(x)=-[\tau_{xy} \text{ in } (13)]_{y=1}\text{: odd function}. \end{aligned}\right\} \tag{14}$$

Let χ_{2r+1} be the biharmonic function which is resolved as follows:

$$\chi_{2r+1}=\chi'+\chi'',$$

where χ' is a function satisfying the boundary condition

$$\sigma_y=\phi(x),\ \tau_{xy}=0 \quad \text{on} \quad y=1, \tag{15}$$

χ'' is a function satisfying the boundary condition

$$\sigma_y=0,\ \tau_{xy}=\psi(x)-\overline{\psi}(x) \quad \text{on} \quad y=1.$$

Consider the function

$$\chi'=\int_0^\infty (A'_u uy \sinh uy+B'_u \cosh uy)\cos ux\, du\ , \tag{16}$$

where A'_u, B'_u are arbitrary constants. Equation (16) is even in both x and y and satisfies the equation

$$\nabla^4\chi'=0.$$

It will satisfy the condition for zero shear on the edge $y=1$ if

$$B'_u=-\frac{\sinh u+u\cosh u}{\sinh u}\,A'_u\,.$$

Representing B'_u in (16) by A'_u,

$$\left.\begin{aligned} &\chi'=\int_0^\infty [(\sinh u+u\cosh u)\cosh uy-uy\sinh u\sinh uy]\,A'_u f(u)\cos ux\, du, \\ \text{where} \qquad &f(u)=-1/\sinh u\,. \end{aligned}\right\} \tag{17}$$

Using (15),

$$\phi(x)=-\frac{1}{2}\int_0^\infty u^2(\sinh 2u+2u)A'_u f(u)\cos ux\, du\,. \tag{18}$$

Inverse sine transformation of this equation is

$$A'_u f(u)=-\frac{4}{\pi u^2(\sinh 2u+2u)}\int_0^\infty \phi(w)\cos uw\, dw. \tag{19}$$

Substituting (19) into (17), we obtain

$$\chi'=\frac{4}{\pi}\int_0^\infty \frac{uysS-(s+uc)C}{u^2\Sigma}\cos ux\, du\int_0^\infty \phi(w)\cos uw\, dw, \tag{20}$$

where

$$s=\sinh u,\ S=\sinh uy,\ c=\cosh u,\ C=\cosh uy,\ \Sigma=\sinh 2u+2u. \tag{21}$$

Let χ'' be the function, which has precisely the same form as χ'.

$$\chi''=\int_0^\infty (A''_u uy \sinh uy + B''_u \cosh uy)\cos ux\, du, \tag{22}$$

where A''_u, B''_u are arbitrary constants. It will satisfy the condition for zero normal stress on the edge $y=1$ if

$$B''_u=-\frac{u\sinh u}{\cosh u}A''_u.$$

Representing B''_u in (22) by A''_u,

$$\left.\begin{aligned}\chi''&=\int_0^\infty(\sinh u\cosh uy-y\cosh u\sinh uy)A''_u f'(u)\cos ux\, du,\\ &\text{where}\\ f'(u)&=-u/\cosh u.\end{aligned}\right\} \tag{23}$$

Using (15),

$$\psi(x)-\overline{\psi}(x)=-\frac{1}{2}\int_0^\infty u(\sinh 2u+2u)A''_u f'(u)\sin ux\, du. \tag{24}$$

Then, $A''_u f'(u)$ is given by

$$A''_u f'(u)=-\frac{4}{\pi}\int_0^\infty \frac{\psi(w)-\overline{\psi}(w)}{u(\sinh 2u+2u)}\sin uw dw. \tag{25}$$

Substituting (25) into (23),

$$\chi''=\frac{4}{\pi}\int_0^\infty \frac{ycS-sC}{u\Sigma}\cos uxdu\int_0^\infty[\psi(w)-\overline{\psi}(w)]\sin uwdw, \tag{26}$$

where $\overline{\psi}(w)$ is undetermined at this stage.

In order that χ_{2r+1} may be the required solution, we must give $\phi(x)$ and $\psi(x)$ the following values, derived from (12) and (13) by putting $\rho^2=1+x^2$;

$$\left.\begin{aligned}\phi(x)&=\frac{2E_2+D_0}{1+x^2}\cos 2\theta+2\sum_{n=1}^\infty\left\{\frac{(2n+1)(nD_{2n}+E_{2n+2})}{(1+x^2)^{n+1}}+\frac{n(2n-1)E_{2n}}{(1+x^2)^n}\right\}\mathrm{co}\\ \psi(x)&=\frac{D_0}{1+x^2}\sin 2\theta+2\sum_{n=1}^\infty\left\{\frac{n(2n+1)D_{2n}}{(1+x^2)^{n+1}}+\frac{n(2n-1)E_{2n}}{(1+x^2)^n}\right\}\sin(2n+2)\end{aligned}\right. \tag{27}$$

θ being the acute angle defined by the equation

$$\tan\theta=x.$$

In order to determine the value of $\overline{\psi}(x)$, it is necessary to evaluate the strain at the line of connection with flange. By considering the condition, $\sigma_y=0$ on $y=1$, the unit elongation in the x-direction is written by

$$[\varepsilon_x]_{y=1}=\frac{1}{E}[\sigma_x]_{y=1}=\frac{1}{E}\left[\frac{\partial^2\chi_{2r}}{\partial y^2}+\frac{\partial^2\chi'}{\partial y^2}+\frac{\partial^2\chi''}{\partial y^2}\right]_{y=1}. \tag{28}$$

The terms on the right-hand side of (28) may be written as follows. Denoting $\left[\frac{\partial^2 \chi_{2r}}{\partial y^2}\right]_{y=1}$ by $\bar{\phi}(x)$ and using (11), we have

$$\left.\begin{aligned}\bar{\phi}(x)&=\left[\frac{\partial^2 \chi_{2r}}{\partial y^2}\right]_{y=1}\\&=\frac{D_0-2E_2}{1+x^2}\cos 2\theta+2\sum_{n=1}^{\infty}\left\{\frac{(2n+1)(nD_{2n}-E_{2n+2})}{(1+x^2)^{n+1}}+\frac{n(2n-1)E_{2n}}{(1+x^2)^n}\right\}\cos(2n+2)\theta,\\ \text{where}\quad &\theta=\tan^{-1}x\,.\end{aligned}\right\} \tag{29}$$

Let $\bar{f}(u)$ be the Fourier cosine transform of $\bar{\phi}(x)$, [$\bar{\phi}(x)$: even function], i.e.,

$$\bar{\phi}(x)=\int_0^\infty \bar{f}(u)\cos uxdu. \tag{30}$$

Then, $\bar{f}(u)$ is written by

$$\bar{f}(u)=\frac{2}{\pi}\int_0^\infty \bar{\phi}(w)\cos uwdw. \tag{31}$$

Substituting (31) into (30),

$$\left[\frac{\partial^2 \chi_{2r}}{\partial y^2}\right]_{y=1}=\frac{2}{\pi}\int_0^\infty \cos ux\,du\int_0^\infty \bar{\phi}(w)\cos uw\,dw. \tag{32}$$

Using (20),

$$\left[\frac{\partial^2 \chi'}{\partial y^2}\right]_{y=1}=\frac{4}{\pi}\int_0^\infty \frac{sc-u}{\Sigma}\cos ux\,du\int_0^\infty \phi(w)\cos uw\,dw. \tag{33}$$

Using (26),

$$\left[\frac{\partial^2 \chi''}{\partial y^2}\right]_{y=1}=\frac{4}{\pi}\int_0^\infty \frac{2c^2}{\Sigma}\cos ux\,du\int_0^\infty [\psi(w)-\bar{\psi}(w)]\sin uw\,dw. \tag{34}$$

Thus, the value of $[\varepsilon_x]_{y=1}$ is given as follows:

$$\begin{aligned}[\varepsilon_x]_{y=1}=\frac{2}{E\pi}\int_0^\infty\Big[2\frac{sc-u}{\Sigma}\int_0^\infty\phi(w)\cos uwdw+\frac{4c^2}{\Sigma}\int_0^\infty\{\psi(w)\\-\bar{\psi}(w)\}\sin uw\,dw+\int_0^\infty\bar{\phi}(w)\cos uw\,dw\Big]\cos ux\,du,\end{aligned} \tag{35}$$

where $\phi(w)$ and $\psi(w)$ are given by (27), $\bar{\phi}(w)$ is given by (29) and $\bar{\psi}(w)$ is not determined yet.

On the other hand, the unit elongation of the flange on the line of connection, $\bar{\varepsilon}_x$ is given by the equation in such a form as

$$\bar{\varepsilon}_x=\frac{2t_w}{E\pi t_f}\int_0^\infty g(u)\cos ux\,du\int_0^\infty \bar{\psi}(w)\sin uw\,dw, \tag{36}$$

where $g(u)$ is a function of u only and depends on the boundary condition. [The calculation of $g(u)$ is given at **2-3** for typical cases.]

Equating (35) to (36),

$$\left.\begin{aligned}&\int_0^\infty \overline{\psi}(w)\sin uw\,dw=\frac{1}{e^{2u}R}\Big[2(sc-u)\int_0^\infty\phi(w)\cos uw\,dw\\&\qquad\qquad +4c^2\int_0^\infty\psi(w)\sin uw dw+\Sigma\int_0^\infty\overline{\phi}(w)\cos uw\,dw\Big],\\&\text{where}\\&\qquad R=e^{-2u}\Big[\frac{t_w}{t_f}g(u)\Sigma+4c^2\Big].\end{aligned}\right\}\tag{37}$$

Substituting (27) and (29) into (37), and considering

$$\left.\begin{aligned}&\int_0^\infty\frac{\cos 2n\theta\cos uw}{(1+w^2)^n}dw=\int_0^\infty\frac{\sin 2n\theta\sin uw}{(1+w^2)^n}dw=\frac{\pi e^{-u}u^{2n-1}}{2(2n-1)!}\\&\int_0^\infty\frac{\cos(2n+2)\theta\cos uw}{(1+w^2)^n}dw=\int_0^\infty\frac{\sin(2n+2)\theta\sin uw}{(1+w^2)^n}dw=\frac{\pi e^{-u}u^{2n-1}(u-n)}{(2n)!}\end{aligned}\right\},\tag{38}^*$$

we attain to the expression

$$\begin{aligned}\int_0^\infty\overline{\psi}(w)\sin uw\,dw=\frac{\pi ue^{-u}}{R}\Big[&D_0(1+e^{-2u})+\sum_{n=1}^\infty D_{2n}\frac{u^{2n}}{(2n-1)!}(1+e^{-2u})\\&-\sum_{n=0}^\infty E_{2n+2}\frac{2u^{2n}}{(2n)!}\Big\{(n+1)(1+e^{-2u})+2ue^{-2u}\Big\}+\sum_{n=1}^\infty E_{2n}\frac{2u^{2n-1}}{(2n-2)!}(1+e^{-2u})\Big].\end{aligned}\tag{39}$$

Then χ'' in (26), subsequently χ_{2r+1} in (15) is determined; i.e.,

$$\chi_{2r+1}=4\int_0^\infty\frac{yS\cos ux}{\Sigma}(s\Phi+c\Psi)du-4\int_0^\infty\frac{C\cos ux}{\Sigma}\Big\{\frac{s+uc}{u}\Phi+s\Psi\Big\}du,\tag{40}$$

where

$$\left.\begin{aligned}&\Phi=\frac{1}{\pi u}\int_0^\infty\phi(w)\cos uw\,dw\\&\Psi=\frac{1}{\pi u}\int_0^\infty[\psi(w)-\overline{\psi}(w)]\sin uw\,dw\end{aligned}\right\}.\tag{41}$$

Now we will determine the stresses produced by this function at the circumference of the hole, and for this purpose it is convenient to express χ_{2r+1} as a series in ascending powers of ρ. Considering

$$\left.\begin{aligned}&yS\cos ux=\frac{1}{2}\Big[\sum_{n=0}^\infty\frac{2nu^{2n-1}\rho^{2n}}{(2n)!}+\sum_{n=0}^\infty\frac{u^{2n+1}\rho^{2n+2}}{(2n+1)!}\Big]\cos 2n\theta\\&C\cos ux=\sum_{n=0}^\infty\frac{(u\rho)^{2n}}{(2n)!}\cos 2n\theta\end{aligned}\right\},\tag{42}$$

we may now express χ_{2r+1} in the form

$$\chi_{2r+1}=\sum_{n=0}^\infty(L_{2n}^{(r)}+M_{2n}^{(r)}\rho^2)\rho^{2n}\cos 2n\theta,\tag{43}$$

$$\left.\begin{aligned}&L_{2n}^{(r)}=\frac{2}{(2n)!}\int_0^\infty[e^u\{(n-u)(\Phi+\Psi)-\Phi\}-e^{-u}\{(n+u)(\Phi-\Psi)-\Phi\}]\frac{u^{2n-1}}{\Sigma}du\\&M_{2n}^{(r)}=\frac{1}{(2n+1)!}\int_0^\infty[e^u(\Phi+\Psi)+e^{-u}(\Psi-\Phi)]\frac{u^{2n+1}}{\Sigma}du\end{aligned}\right\}.\tag{44}$$

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* See pp. 56 and 57, **Reference** 1).

Restoring the suffix (r) to the coefficients of χ_{2r}, we find

$$\left.\begin{aligned} L_{2n}^{(r)} &= {}^n\alpha_0 D_0^{(r)} + \sum_{p=1}^{\infty}\{{}^n\alpha_p D_{2p}^{(r)} + {}^n\beta_p E_{2p}^{(r)}\} \\ M_{2n}^{(r)} &= {}^n\gamma_0 D_0^{(r)} + \sum_{p=1}^{\infty}\{{}^n\gamma_p D_{2p}^{(r)} + {}^n\delta_p E_{2p}^{(r)}\} \end{aligned}\right\}, \tag{45}$$

$$\left.\begin{aligned} {}^n\alpha_0 &= \frac{1}{(2n)!}\Big[(2n-1)I_{2n-1} - 2I_{2n} + J_{2n-1} \\ &\qquad -2\int_0^\infty \frac{1+e^{-2u}}{R}\{n(1+e^{-2u}) - u(1-e^{-2u})\}\frac{u^{2n-1}}{\Sigma}du\Big] \\ {}^n\alpha_p &= \frac{1}{(2n)!(2p-1)!}\Big[(2n-1)I_{2n+2p-1} - 2I_{2n+2p} + J_{2n+2p-1} \\ &\qquad -2\int_0^\infty \frac{1+e^{-2u}}{R}\{n(1+e^{-2u}) - u(1-e^{-2u})\}\frac{u^{2n+2p-1}}{\Sigma}du\Big] \\ {}^n\beta_p &= \frac{2}{(2n)!(2p-2)!}\Big[2(n+p-1)I_{2n+2p-2} - (2np-n-p+1)I_{2n+2p-3} \\ &\qquad -2I_{2n+2p-1} - (n+p-1)J_{2n+2p-3} \\ &\qquad +2\int_0^\infty \frac{p(1+e^{-2u}) - u(1-e^{-2u})}{R}\{n(1+e^{-2u}) - u(1-e^{-2u})\}\frac{u^{2n+2p-3}}{\Sigma}du\Big] \\ {}^n\gamma_0 &= \frac{1}{(2n+1)!}\Big[I_{2n+1} - \int_0^\infty \frac{(1+e^{-2u})^2}{R}\frac{u^{2n+1}}{\Sigma}du\Big] \\ {}^n\gamma_p &= \frac{1}{(2n+1)!(2p-1)!}\Big[I_{2n+2p+1} - \int_0^\infty \frac{(1+e^{-2u})^2}{R}\frac{u^{2n+2p+1}}{\Sigma}du\Big] \\ {}^n\delta_p &= -\frac{1}{(2n+1)!(2p-2)!}\Big[(2p-1)I_{2n+2p-1} - 2I_{2n+2p} + J_{2n+2p-1} \\ &\qquad -\int_0^\infty \frac{2p(1+e^{-2u})^2 - 2u(1-e^{-4u})}{R}\frac{u^{2n+2p-1}}{\Sigma}du\Big] \end{aligned}\right\}, \tag{46}$$

where

$$\left.\begin{aligned} I_s &= \int_0^\infty \frac{u^s}{\Sigma}du \\ J_s &= \int_0^\infty \frac{u^s}{\Sigma}e^{-2u}du \end{aligned}\right\}. \tag{47)*$$

The terms of the integrals in (46) represent the effects of the flange and vanish $(R\to\infty)$ when the perforated strip has no flange.

The coefficients in χ_{2r+1} are thus determined in terms of those of χ_{2r}. To complete the cycle, we have now to determine the coefficients in χ_{2r+2} in terms of those of χ_{2r+1}. The stresses due to χ_{2r+1} are given immediately by the differentiation of (43) as

$$\left.\begin{aligned} \sigma_\rho &= 2\Big[M_0^{(r)} - \sum_{n=1}^{\infty}\Big\{n(2n-1)L_{2n}^{(r)} + (n-1)(2n+1)M_{2n}^{(r)}\rho^2\Big\}\rho^{2n-2}\cos 2n\theta\Big] \\ \tau_{\rho\theta} &= 2\sum_{n=1}^{\infty}\Big\{n(2n-1)L_{2n}^{(r)} + n(2n+1)M_{2n}^{(r)}\rho^2\Big\}\rho^{2n-2}\sin 2n\theta \end{aligned}\right\}. \tag{48}$$

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* The values of I_s and J_s for $S\leq 20$ have been obtained by Howland [See p. 67, **Reference** 1)].

$$\sigma_\theta=2\Big[M_0^{(r)}+\sum_{n=1}^{\infty}\Big\{n(2n-1)L_{2n}^{(r)}+(n+1)(2n+1)M_{2n}^{(r)}\rho^2\Big\}\rho^{2n-2}\cos 2n\theta\Big]$$

The first two of these must be cancelled at the edge of the hole by the stresses due to χ_{2r+2}. If we express χ_{2r+2} in such a form as χ_{2r};

$$\chi_{2r+2}=-D_0^{(r+1)}\log\rho+\sum_{n=1}^{\infty}\Big\{\frac{D_{2n}^{(r+1)}}{\rho^{2n}}+\frac{E_{2n}^{(r+1)}}{\rho^{2n-2}}\Big\}\cos 2n\theta, \tag{49}$$

the corresponding stresses will be given by adding the suffix $(r+1)$ to the coefficients in equations (8) and (9). Putting $\rho=\lambda$ and equating the coefficients to the negatives of those in (48), we obtain the following equations:

$$\left.\begin{aligned} &D_0^{(r+1)}=2M_0^{(r)}\lambda^2\\ &D_{2n}^{(r+1)}=\lambda^{4n}\{(2n-1)L_{2n}^{(r)}+2n\lambda^2M_{2n}^{(r)}\}\\ &E_{2n}^{(r+1)}=-\lambda^{4n-2}\{2nL_{2n}^{(r)}+(2n+1)\lambda^2M_{2n}^{(r)}\}\end{aligned}\right\}. \tag{50}$$

From (45), (46) and (50), we can determine the coefficients in terms of those at the stage of the preceding approximation.

2-2 Determination of the Stresses

To determine the stress function of the perforated strip under the uniform tensile stress T, we start with the stress function, χ_0', of the perfect strip under tension.

$$\chi_0'=T\rho^2(1+\cos2\theta)/4 \tag{51}$$

satisfies the definition. And the stress function χ_0 which produces on the edge of the hole stresses cancelling the values of σ_ρ and $\tau_{\rho\theta}$ due to χ_0' are obtained by replacing the coefficients in (7) by the following values:

$$\left.\begin{aligned}&D_0^{(0)}=T\lambda^2/2,\ D_2^{(0)}=T\lambda^4/4,\ E_2^{(0)}=-T\lambda^2/2\\ &\text{and all the other coefficients are zero.}\end{aligned}\right\} \tag{52}$$

Substituting (52) into (45), the coefficients in χ_1 are given by

$$\left.\begin{aligned}&L_{2n}^{(0)}=T\lambda^2\{({}^n\alpha_0-{}^n\beta_1)/2+{}^n\alpha_1\lambda^2/4\}\\ &M_{2n}^{(0)}=T\lambda^2\{({}^n\gamma_0-{}^n\delta_1)/2+{}^n\gamma_1\lambda^2/4\}\end{aligned}\right\}. \tag{53}$$

From these it is easy to calculate the values of the coefficients, $D_0^{(1)}$, $D_{2n}^{(1)}$ and $E_{2n}^{(1)}$, of χ_2 by using (50). Proceeding in this way, we find $L_{2n}^{(r-1)}$ and $M_{2n}^{(r-1)}$ from $D_0^{(r-1)}$, $D_{2n}^{(r-1)}$ and $E_{2n}^{(r-1)}$ by using (45), and further $D_0^{(r)}$, $D_{2n}^{(r)}$ and $E_{2n}^{(r)}$ from $L_{2n}^{(r-1)}$ and $M_{2n}^{(r-1)}$ by using (50).

The final value of χ is given by

$$\chi=T\rho^2(1+\cos2\theta)/4+T\Big[-d_0\log\rho+m_0\rho^2+\sum_{n=1}^{\infty}\Big\{\frac{d_{2n}}{\rho^{2n}}+\frac{e_{2n}}{\rho^{2n-2}}+(l_{2n}+m_{2n}\rho^2)\rho^{2n}\Big\}\cos 2n\theta\Big], \tag{54}$$

where

$$\left.\begin{aligned}&d_0=\sum_r D_0^{(r)}/T,\ d_{2n}=\sum_r D_{2n}^{(r)}/T,\ e_{2n}=\sum_r E_{2n}^{(r)}/T\\&l_{2n}=\sum_r L_{2n}^{(r)}/T,\ m_0=\sum_r M_0^{(r)}/T,\ m_{2n}=\sum_r M_{2n}^{(r)}/T.\end{aligned}\right\}\tag{55}$$

The stresses are

$$\left.\begin{aligned}\sigma_\rho=T\Big[&\frac{1}{2}(1-\cos 2\theta)+2m_0-\frac{d_0}{\rho^2}-2\sum_{n=1}^{\infty}\Big\{\frac{n(2n+1)d_{2n}}{\rho^{2n+2}}+\frac{(n+1)(2n-1)e_{2n}}{\rho^{2n}}\\&+n(2n-1)l_{2n}\rho^{2n-2}+(n-1)(2n+1)m_{2n}\rho^{2n}\Big\}\cos 2n\theta\Big]\\\sigma_\theta=T\Big[&\frac{1}{2}(1+\cos 2\theta)+2m_0+\frac{d_0}{\rho^2}+2\sum_{n=1}^{\infty}\Big\{\frac{n(2n+1)d_{2n}}{\rho^{2n+2}}+\frac{(n-1)(2n-1)e_{2n}}{\rho^{2n}}\\&+n(2n-1)l_{2n}\rho^{2n-2}+(n+1)(2n+1)m_{2n}\rho^{2n}\Big\}\cos 2n\theta\\\tau_{\rho\theta}=T\Big[&\frac{1}{2}\sin 2\theta+2\sum_{n=1}^{\infty}\Big\{n(2n-1)\Big(l_{2n}\rho^{2n-2}-\frac{e_{2n}}{\rho^{2n}}\Big)\\&+n(2n+1)\Big(m_{2n}\rho^{2n}-\frac{d_{2n}}{\rho^{2n+2}}\Big)\Big\}\sin 2n\theta\Big]\end{aligned}\right\}.\tag{56}$$

The circumferential stress on the edge of the hole is obtained, by putting $\rho=\lambda$ in the second of equations (56),

$$\left.\begin{aligned}&\sigma_\theta=2T\sum_{n=0}^{\infty}P_{2n}\cos 2n\theta\\&P_0=1/4+m_0+d_0/(2\lambda^2)\\&P_2=1/4+3d_2/\lambda^4+l_2+6m_2\lambda^2\\&P_4=10d_4/\lambda^6+3e_4/\lambda^4+6l_4\lambda^2+15m_4\lambda^4\\&P_6=21d_6/\lambda^8+10e_6/\lambda^6+15l_6\lambda^4+28m_6\lambda^6\\&P_8=36d_8/\lambda^{10}+21e_8/\lambda^8+28l_8\lambda^6+45m_8\lambda^8\\&\cdots\cdots\cdots\cdots\end{aligned}\right\}.\tag{57}$$

2-3 Calculation of $g(u)$ and R

We will determine $g(u)$ and R for the typical cases as shown in Fig. 2.

The stress function of the flange, $\bar{\chi}$, which satisfies

$$\frac{\partial^4\bar{\chi}}{\partial x^4}+2\frac{\partial^4\bar{\chi}}{\partial x^2\partial z^2}+\frac{\partial^4\bar{\chi}}{\partial z^4}=0,\tag{58}$$

is given by

$$\bar{\chi}=\int_0^\infty[(\bar{A}_u'+\bar{C}_u'uz)\cosh uz+(\bar{B}_u'+\bar{D}_u'uz)\sinh uz]\cos uxdu,\tag{59}$$

where $\bar{A}_u'$, $\bar{B}_u'$, $\bar{C}_u'$ and $\bar{D}_u'$ are arbitrary constants, to be determined from the boundary conditions.

[Case-I] H-Beam

Boundary condition:

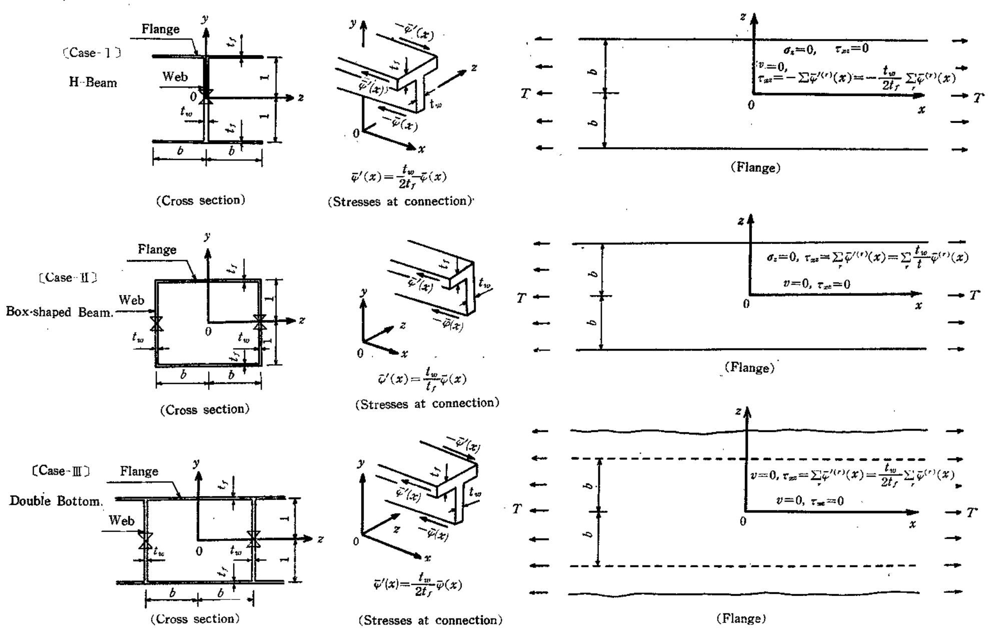

Fig. 2. Typical structures and these boundary conditions

$$\left.\begin{array}{ll} v=0,\ \tau_{xz}=-\overline{\psi}'(x)=-\dfrac{t_w}{2t_f}\overline{\psi}(x) & \text{on}\quad z=0, \\ \sigma_z=0,\quad \tau_{xz}=0 & \text{on}\quad z=b \end{array}\right\} \qquad (60)$$

where v is the displacement in the z-direction.

From the condition $v=0$ on $z=0$,

$$\frac{\overline{B}'_u}{\overline{C}'_u}=\frac{1-\nu}{1+\nu}\equiv j,\qquad (\nu:\ \text{Poisson's ratio}).$$

From the condition $\sigma_z=0$, $\tau_{xz}=0$ on $z=b$,

$$\overline{C}'_u=-\frac{ub+\bar{s}\bar{c}}{u^2b^2+j\bar{s}^2}\overline{A}'_u,\qquad \overline{D}'_u=\frac{j+\bar{c}^2}{u^2b^2+j\bar{s}^2}\overline{A}'_u, \qquad (61)$$

where $\bar{s}=\sinh ub,\quad \bar{c}=\cosh ub.$

Substituting (61) into (59), we obtain the stress function containing $\overline{A}'_u$ alone.

$$\overline{\chi}=\int_0^\infty\left[(u^2b^2+j\bar{s}^2)\overline{C}-\frac{\overline{\Sigma}}{2}(uz\overline{C}+j\overline{S})+(j+\bar{c}^2)uz\overline{S}\right]\overline{A}'_u\overline{f}'(u)\cos ux\,du, \qquad (62)$$

where

$$\overline{C}=\cosh uz,\quad \overline{S}=\sinh uz,\quad \overline{\Sigma}=\sinh 2ub+2ub$$

$$\overline{f}'(u)=\frac{1}{u^2b^2+j\bar{s}^2}$$

From the condition $[\tau_{xz}]_{z=0}=-\left[\dfrac{\partial^2\overline{\chi}}{\partial x\partial z}\right]_{z=0}=-\dfrac{t_w}{2t_f}\overline{\psi}(x)$, it follows that

$$\bar{A}'_u\bar{f}'(u)=\frac{2t_w}{\pi(1+j)t_f}\int_0^\infty\frac{\bar{\psi}(w)}{u^2\bar{\Sigma}}\sin uw\,dw. \tag{63}$$

Then,

$$\bar{\chi}=\frac{2t_w}{\pi(1+j)t_f}\int_0^\infty\left[\frac{(u^2b^2+j\bar{s}^2)\bar{C}}{u^2\bar{\Sigma}}-\frac{1}{2}\frac{uz\bar{C}+j\bar{S}}{u^2}+\frac{(j+\bar{c}^2)z\bar{S}}{u\bar{\Sigma}}\right]\cos uxdu\int_0^\infty\bar{\psi}(w)\sin uw\,dw. \tag{64}$$

Subsequently, $g(u)$ in (36) and R in (37) are

$$g(u)=\frac{1}{(1+j)\bar{\Sigma}}[(u^2b^2+j\bar{s}^2)(1+\nu)+2(j+\bar{c}^2)], \tag{65}$$

$$R=\frac{t_w}{2t_f}\frac{\Sigma}{\bar{\Sigma}}e^{-2u}\left[(1+\nu)^2u^2b^2+\frac{(1+\nu)(3-\nu)}{4}(e^{2ub}+e^{-2ub})+\frac{5-2\nu+\nu^2}{2}\right]+(1+e^{-2u})^2. \tag{66}$$

[Case-II] Box-shaped Beam

Boundary condition:

$$\left.\begin{aligned}&v=0,\quad \tau_{xz}=0 && \text{on}\quad z=0,\\ &\sigma_z=0,\quad \tau_{xz}=\bar{\psi}'(x)=\frac{t_w}{t_f}\bar{\psi}(x) && \text{on}\quad z=b.\end{aligned}\right\} \tag{67}$$

From the condition $v=0$ on $z=0$,

$$\bar{B}'_u/\bar{C}'_u=j.$$

From the condition $\tau_{xz}=0$ on $z=0$,

$$\bar{C}'_u=0.$$

From the condition $\sigma_z=0$ on $z=b$,

$$\bar{D}'_u=-\frac{\bar{c}}{ub\bar{s}}\bar{A}'_u. \tag{68}$$

Substituting (68) into (59), we obtain

$$\bar{\chi}=\int_0^\infty[ub\bar{s}\bar{C}-uz\bar{c}\bar{S}]\,\bar{A}'_u\bar{f}'(u)\cos ux\,du,$$

where $\quad \bar{f}'(u)=\dfrac{1}{ub\bar{s}}\quad .$ (69)

From the condition $[\tau_{xz}]_{z=b}=-\left[\dfrac{\partial^2\bar{\chi}}{\partial x\partial z}\right]_{z=b}=\dfrac{t_w}{t_f}\bar{\psi}(x)$,

$$\bar{A}'_u\bar{f}'(u)=-\frac{4t_w}{\pi t_f}\int_0^\infty\frac{\bar{\psi}(w)}{u^2\bar{\Sigma}}\sin uw\,dw. \tag{70}$$

Then,

$$\bar{\chi}=-\frac{4t_w}{\pi t_f}\int_0^\infty\frac{b\bar{s}\bar{C}-z\bar{c}\bar{S}}{u\bar{\Sigma}}\cos ux\,du\int_0^\infty\bar{\psi}(w)\sin uw\,dw. \tag{71}$$

Subsequently,

$$g(u)=4\frac{\bar{c}^2}{\Sigma}, \tag{72}$$

$$R=e^{-2u(1-b)}(1+e^{-2ub})^2\frac{t_w\Sigma}{t_f\bar{\Sigma}}+(1+e^{-2u})^2. \tag{73}$$

[Case-III] Double Bottom

Boundary condition:

$$\left.\begin{array}{lll} v=0, & \tau_{xz}=0 & \text{on } z=0, \\ v=0, & \tau_{xz}=\bar{\psi}'(x)=\dfrac{t_w}{2t_f}\bar{\psi}(x) & \text{on } z=b. \end{array}\right\} \tag{74}$$

From the condition $v=0$, $\tau_{xz}=0$ on $z=0$,

$$\bar{B}'_u/\bar{C}'_u=j,\quad \bar{C}'_u=0.$$

From the condition $v=0$ on $z=b$,

$$\bar{D}'_u=\frac{\bar{s}}{j\bar{s}-ub\bar{c}}\bar{A}'_u. \tag{75}$$

Substituting (75) into (59), we obtain

$$\bar{\chi}=\int_0^\infty[(js-ub\bar{c})\bar{C}+uzs\bar{S}]\bar{A}'_u\bar{f}'(u)\cos ux\,du,$$

where $$\bar{f}'(u)=\frac{1}{j\bar{s}-ub\bar{c}}. \tag{76}$$

From the condition $[\tau_{xz}]_{z=b}=-\left[\dfrac{\partial^2\bar{\chi}}{\partial x\partial z}\right]_{z=b}=\dfrac{t_w}{2t_f}\bar{\psi}(x)$,

$$\bar{A}'_u\bar{f}'(u)=\frac{t_w}{\pi(1+j)t_f}\int_0^\infty\frac{\bar{\psi}(w)}{u^2\bar{s}^2}\sin uw\,dw. \tag{77}$$

Substituting (77) into (76),

$$\bar{\chi}=\frac{t_w}{\pi(1+j)t_f}\int_0^\infty\frac{(js-ubc)\bar{C}+uzs\bar{S}}{u^2\bar{s}^2}\cos ux\,du\int_0^\infty\bar{\psi}(w)\sin uw\,dw. \tag{78}$$

Subsequently,

$$g(u)=\frac{1}{2(1+j)}\frac{(3-\nu)\bar{s}\bar{c}-(1+\nu)ub}{\bar{s}^2}, \tag{79}$$

$$R=\frac{t_we^{-2u}\Sigma}{2t_f(1-e^{-2ub})^2}\left[\frac{(1+\nu)(3-\nu)}{2}(1-e^{-4ub})-2(1+\nu)^2ube^{-2ub}\right]+(1+e^{-2u})^2. \tag{80}$$

3. Numerical calculation

Numerical calculation is performed for the case-II. The value of stress concentration factor $\sigma_{\theta max.}/T$ is shown against λ, together with the one of the perforated strip without flange. (Fig. 3)

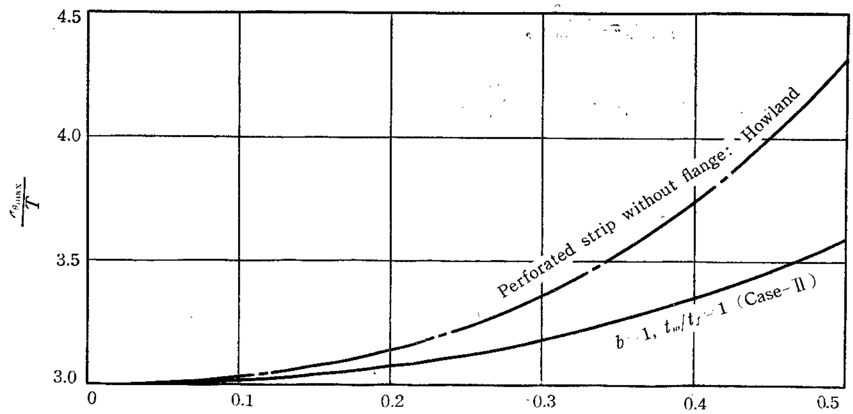

Fig. 3. The stress concentration factors for the case-II

The value of $\sigma_{\theta max.}/T$ increases with λ in both cases, but the ratio of the increment is smaller in the case-II. While the ratio of the increment depends on the scantling of the flange, the effects of the flange seem to be considerable. (The numerical calculation is now being continued.)

4. Conclusion

The formulae for the stresses on the edge of the hole in the flanged strip under tension are obtained for the typical three cases. And the results of the numerical calculation for the box-shaped beam are shown to illustrate the effects of the flange on the decrement of the stress concentration. (The numerical calculation is now being continued.)

References

1) R.C.J., Howland, Phil. Trans. of the Roy. Soc. Ser. A, vol. 229 (1930).