On the Velocity Distribution at the Inlet in a Circular Pipe

On the Laminar Velocity Distribution at the Inlet in a Circular Pipe

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(Received January 28, 1957)

Abstract

In general it is assumed that the laminar velocity profile at the inlet in a horizontal circular pipe is parabolic in the boundary layer. In this paper the correctness of this assumption was proved by means of measured values of the core velocity and the pressure drop.

1. Nomenclatures

u_0 = mean velocity
u' = axial velocity in core flow
u = axial velocity in boundary layer
v = radial velocity in boundary layer
R = inside radius of pipe
δ = thickness of boundary layer
x = distance from pipe entrance in axial direction
r = radius from pipe center
p = static pressure in fluid
μ = coefficient of viscosity of fluid
ρ = density of fluid
ν = kinematic viscosity of fluid
$h = p/\rho g$ = pressure head
g = gravitational acceleration
$Re = Ru_0/\nu$ = Reynolds' number
$z = x/R,\quad t = r/R,\quad y = \delta/R,\quad \eta = (1-t)/2y,$
$$\Lambda = \frac{dh}{(u_0^2/2g)(dz/Re)}\frac{u_0}{u'}.$$

2. Velocity Distribution

It was assumed that the axial velocity distribution in boundary layer was expressed in the following form.

$$u = R^3(1-t)^3 a + R^2(1-t)^2 b + R(1-t)\, c + d. \tag{1}$$

Navier-Stokes' equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right). \tag{2}$$

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Boundary conditions:

$$\begin{array}{llll} \text{(a)} & u=v=0 & \text{for} & t=1 \\ \text{(b)} & u=u' & \text{for} & t=1-y \\ \text{(c)} & \dfrac{\partial u}{\partial t}=0 & \text{for} & t=1-y \end{array}$$

Substituting the conditions (a), (b) and (c) for Eq. (1),

$$d=0\,, \tag{3}$$

$$R^3y^3a+R^2y^2b+Ryc+d=u'\,, \tag{4}$$

$$3R^2y^2a+2Ryb+c=0\,. \tag{5}$$

From Eq. (1) for $t=1$

$$\frac{\partial u}{\partial t}=-Rc\,, \qquad \frac{\partial^2 u}{\partial t^2}=2R^2b\,. \tag{6}$$

Substituting the boundary condition (a) and Eq. (6) for Eq. (2)

$$-\frac{dp}{dz}+\mu(2Rb-c)=0\,. \tag{7}$$

From Eq. (3), (4), (5) and (7)

$$a=-\frac{Ry^2(dp/dz)+2\mu(1+y)\,u'}{\mu R^3(4+y)\,y^3}\,, \tag{8}$$

$$b=\frac{2Ry(dp/dz)+3\mu u'}{\mu R^2(4+y)\,y}\,, \tag{9}$$

$$c=\frac{6\mu u'-Ry^2(dp/dz)}{\mu R(4+y)\,y}\,. \tag{10}$$

Substituting these equations (3), (8), (9) and (10) for Eq. (1) and arranging

$$\frac{u}{u'}=\frac{4}{4+y}\left[\left(-\frac{y^2}{4}\Lambda+3\right)\eta+(\eta\Lambda+3)\,y\eta^2+\{-y^2\Lambda-4(1+y)\}\right]. \tag{11}$$

From the continuity of flow

$$\pi R^2u_0=\pi R^2(1-y)^2\,u'+2\pi R^2\int_{1-y}^{1}utdt\,. \tag{12}$$

The left side of this formula means the discharge at the pipe entrance where uniform distribution of velocity u_0 is held, while the first term of the right side shows the discharge in core flow and the second term shows that in boundary layer. Substituting the Eq. (11) for Eq. (12),

$$\frac{u_0}{u'}=\frac{1}{60(4+y)}\{(2y-5)\,y^3\Lambda+6(3y^3-2y^2-20y+40)\}\,. \tag{13}$$

Dividing the both sides of Eq. (13) by u_0/u', the following equation is obtained.

$$(2y-5)\,y^3\frac{dh}{(u_0^2/2g)\,(dz/Re)}+6(3y^3-2y^2-20y+40)\frac{u'}{u_0}-60(4+y)=0\,. \tag{14}$$

3. Pressure Drop

The experiment has been made under the following conditions.

Pipe: brass (inside radius R=11.475 mm, 8.500 mm and 5.540 mm)

Fluid: water (5~30°C)

Fig. 1 indicates the sketch of the equipment for the experiment. Water in a reservoir a is fed to an upper tank c by means of a gear pump b. The over-flow d is set at the upper tank and holds the level of water constant. Water is led from the upper tank to a lower tank e which is closed up tightly and arranges the dividing plate g in order to prevent the turbulence of water at the entrance of brass pipe f. The entry of pipe is so sufficiently rounded (radius of curvature is about 50 mm) that the velocity profile at the

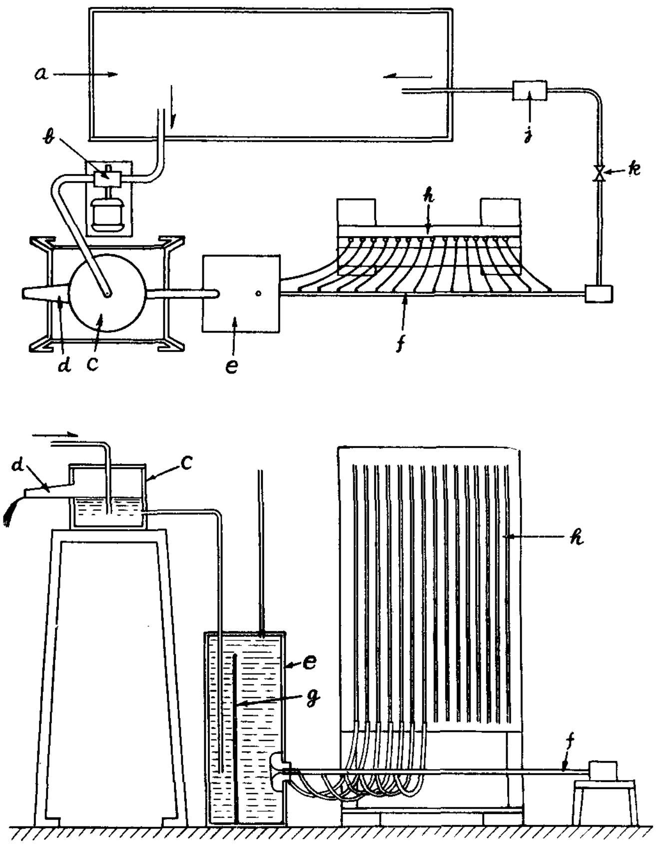

Fig. 1

entrance (x=0) is uniform. These brass pipes are 2 meters long, whose inside surfaces are smooth. Static pressure in these pipes is measured with many manometers h. The positions of measurement were chosen at the intervals of 10 cm from the entrance (x=0). Water returns to the reservoir a passing through a flow meter j after the pipe. Discharge was adjusted by a cock k and calculated by means of precisely measured weight of water which enters in a vessel before reservoir, and time flowed out. Flow meter j was used only for the standard in adjustment of discharge. Utillized Reynolds' number ranged from about 200 to 1000. Mean velocity u_0 was calculated by using of the discharge and the

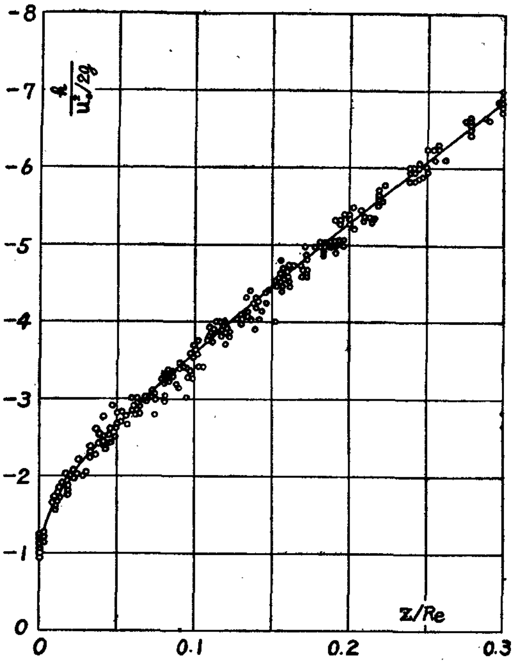

Fig. 2

sectional area of pipe. Fig. 2 shows the results of this experiment.[1]

4. Core Velocity

Core velocity in the pipe was measured with a Pitot tube inserted into the brass pipe of Fig. 1. This Pitot tube, whose structure is indicated in Fig. 3, can be moved up to any position in the pipe where it corresponds with the manometers h, so that static pressure and dynamic pressure may be at the same time measured, then core velocity was given from these. Fig. 4 shows this measured values.[2] The measurement was done for the two brass pipes (R=11.475 mm and 8.500 mm).

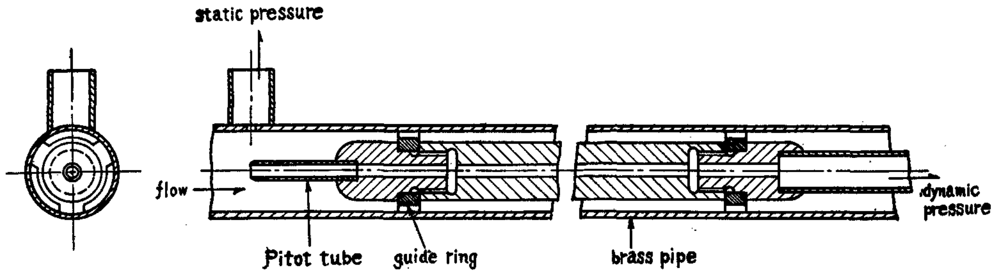

Fig. 3

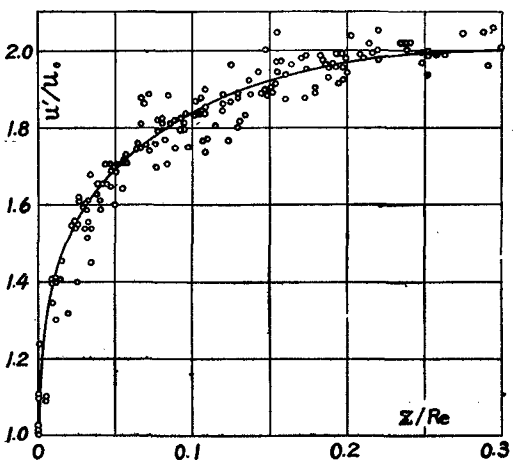

Fig. 4

5. Utilization of Measured Values

The values of u'/u_0 and $dh/(u_0^2/2g)(dz/Re)$ are obtained from the measured values in Fig. 2 and 4 as shown in Table 1, therefore the values of Λ are given too. If the values of u'/u_0 and $dh/(u_0^2/2g)(dz/Re)$ in Table 1 are substituted for Eq. (14), the corresponding values of y are obtained. In Table 1 the values of y from Eq. (14) are shown with the calculated values of y in the previous report.[2] Fig. 5 indicates the relation between Λ and y which has been got from Eq. (14).

Table 1

z/Re	$\frac{dh}{(u_0^2/2g)(dz/Re)}$	u'/u_0	Λ	y from Eq. (14)	y Calculated[2]
0	$-\infty$	1.000	$-\infty$	0	0
0.01	−34.50	1.405	−24.56	0.4932	0.494
0.02	−26.75	1.520	−17.60	0.6049	0.604
0.03	−23.60	1.595	−14.80	0.6728	0.673
0.04	−21.85	1.650	−13.24	0.7207	0.726
0.05	−20.65	1.695	−12.18	0.7586	0.751
0.06	−19.80	1.735	−11.41	0.7923	0.800
0.07	−19.15	1.770	−10.82	0.8214	0.827
0.08	−18.70	1.795	−10.42	0.8416	0.851
0.09	−18.30	1.820	−10.05	0.8618	0.872
0.10	−17.90	1.845	− 9.70	0.8813	0.889
0.11	−17.60	1.865	− 9.44	0.8968	0.904
0.12	−17.40	1.880	− 9.26	0.9086	0.916
0.13	−17.20	1.895	− 9.08	0.9202	0.928
0.14	−17.00	1.910	− 8.90	0.9316	0.938
0.15	−16.85	1.925	− 8.75	0.9438	0.947
0.16	−16.70	1.935	− 8.63	0.9508	0.954
0.17	−16.60	1.945	− 8.53	0.9587	0.961
0.18	−16.45	1.955	− 8.41	0.9655	0.967
0.19	−16.35	1.965	− 8.32	0.9733	0.971
0.20	−16.30	1.970	− 8.27	0.9772	0.975
0.21	−16.25	1.975	− 8.23	0.9810	0.979
0.22	−16.20	1.980	− 8.18	0.9849	0.982
0.23	−16.15	1.985	− 8.14	0.9887	0.985
0.24	−16.10	1.990	− 8.09	0.9925	0.987
0.25	−16.05	1.995	− 8.05	0.9962	0.989
0.26	−16.00	2.000	− 8.00	1.0000	0.990

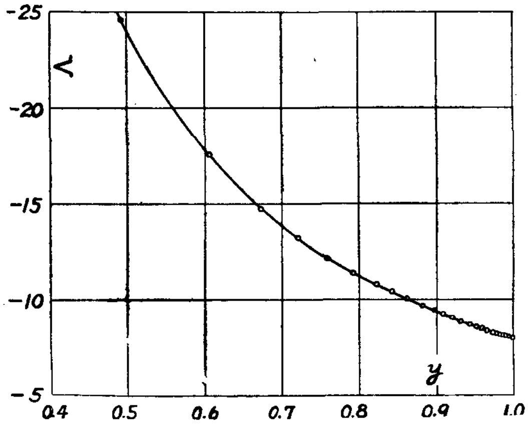

Fig. 5

Table 2

Y	y	Λ	$\Delta\Lambda$	$\Delta^2\Lambda$
1.0	1.000	− 8.00		
			1.24	
1.1	0.909	− 9.24		0.08
			1.32	
1.2	0.833	−10.56		0.08
			1.40	
1.3	0.769	−11.96		0.08
			1.48	
1.4	0.714	−13.44		0.08
			1.56	
1.5	0.667	−15.00		0.08
			1.64	
1.6	0.625	−16.64		0.08
			1.72	
1.7	0.588	−18.36		0.08
			1.80	
1.8	0.556	−20.16		0.08
			1.88	
1.9	0.526	−22.04		0.08
			1.96	
2.0	0.500	−24.00		

When Y which is equal to $1/y$ is taken at equal intervals and the corresponding values of Λ are picked up from Fig. 5, the differences of Λ, $\Delta\Lambda$ and $\Delta^2\Lambda$, yield as shown in Table 2 and $\Delta^2\Lambda$ results in constant. Therefore, we can put Λ in the following form.

$$\Lambda = k_0 + k_1 Y + k_2 Y^2 . \tag{15}$$

The coefficients k_0, k_1 and k_2 which satisfy the values of Table 2, are as follows.

$$\left. \begin{aligned} k_0 &= 0 , \\ k_1 &= k_2 = -4 . \end{aligned} \right\} \tag{16}$$

Namely,

$$\Lambda = -4(1+y)/y^2 . \tag{17}$$

Substituting Eq. (17) for Eq. (11),

$$u/u' = 4(\eta - \eta^2) . \tag{18}$$

In other words the velocity profile in boundary layer becomes to parabolic form in the range $y = 0.5 \sim 1$.

6. Velocity Distribution Near Entrance

Near the pipe entrance, $y < 0.5$, the errors of measured values are so large, that the above-mentioned method can be hardly applied. Now we assume that Eq. (18) is available for $y < 0.5$ also. Substituting Eq. (18) for Eq. (12),

$$u'/u_0 = 6/(6 - 4y + y^2) . \tag{19}$$

From the definition of Λ,

$$\frac{h}{u_0^2/2g} = \int_0^{z/Re} \Lambda \frac{u'}{u_0} \frac{dz}{Re} + C . \tag{20}$$

By substitution of Eq. (17) and (19) for this,

$$\frac{h}{u_0^2/2g} = -\int_0^{z/Re} \frac{24(1+y)}{(6-4y+y^2)\, y^2}\, d\left(\frac{z}{Re}\right) + C . \tag{21}$$

Since, at $z/Re = 0$, this equation becomes to infinitive and can not be

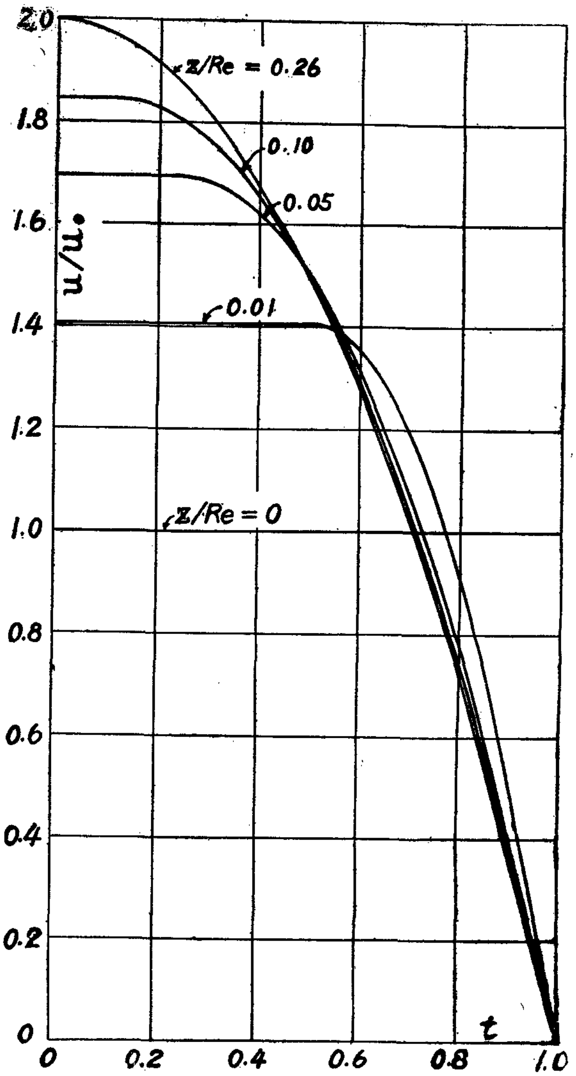

Fig. 6

Table 3

z/Re	$\frac{h}{u_0^2/2g}$
0.014	−1.8695
0.012	−1.8073
0.010	−1.7410
0.008	−1.6690
0.006	−1.5889
0.004	−1.4958
0.002	−1.3744
0	−1.0000

integrated, the value of $h/(u_0^2/2g)$ at $z/Re=0.01$ was found from Fig. 2 and from this position the integration was done numerically up to $z/Re=0$. The integral constant C is determined from Fig. 2. Thus,

$$\frac{h}{u_0^2/2g}=\int_{z/Re}^{0.01}\frac{24(1+y)}{(6-4y+y^2)\,y^2}\,d\left(\frac{z}{Re}\right)-1.741\,. \quad (22)$$

In this integration the relation between y and z/Re was quoted from our previous report.[2] But, since $z/Re=0$ is singular point, we defined $h/(u_0^2/2g)=-1$ for this point. This calculated result is shown in Table 3 and has good agreement with Fig. 2. Therefore, we may consider that the velocity distribution in boundary layer near the entrance corresponding to the range $y<0.5$ is parabolic too.

7. Conclusions

Fig. 6 shows the velocity profile at the cross-section $z/Re=0.01$, 0.05, 0.10 and 0.26. Besides, Fig. 7 shows the relation between u/u_0 and inlet length z/Re for $r/R=0$, 0.2, 0.4, 0.6, 0.7, 0.8 and 0.9. In this figure the dotted lines are the measured results by Nikuradse[3] and the two chain lines show the calculated values by Boussinesq[4] and Schiller[5].

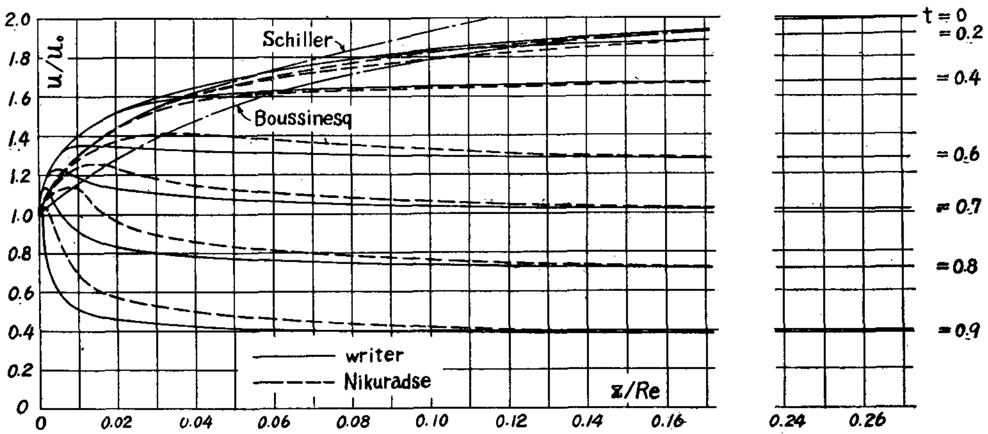

Fig. 7

The conclusions in this paper are as follows.

(1) The velocity distribution in laminar boundary layer at the inlet is parabolic as shown in Eq. (18) except near the entrance.

(2) The velocity profile in boundary layer, very near the pipe entrance corresponding to the range $z/Re<0.01$, also may be considered as parabolic form.

(3) The relation between the velocity distribution and the inlet length is shown in Fig. 7.

We can expect that the calculating method described in this paper can be applied for the velocity distribution in non-isothermal flow by means of measured values of core velocity and pressure drop.

References

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