The Dynamic Young's Modulus in Compacted Clayey Soil at High Pressures and the Effects of Radial Inertia and Added Mass

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Introduction

The experimental and analytical results presented in this paper were obtained in course of an investigation into the factors controlling the dynamic Young's modulus of compacted clayey soil at high pressures under various conditions of moisture content, degree of saturation, void ratio and density. In a study of the effects of radial inertia and attached weight in longitudinal vibration of a cylindrical rod, analytical results showed the necessity of the significant correction to the theory for the velocity of wave propagation and frequency of vibration. An analytical solution for the vibration of a cylindrical rod was presented using the three dimensional elasticity theory by L. Pochhammer.[1] However, this solution is very complex and it may not be expected that we apply the solution to the measurement of dynamic Young's modulus of clayey soils.

Therefore, in this paper, we introduced the one-dimensional wave equation including a radial inertia term according to the energy method[2] and showed an analytical solution of this equation which satisfy the boundary conditions of the resonant test.

As the other solutions[3] that take internal damping of the soil into account have been shown that the elastic solution are satisfactory within the range of damping developed for small-strain amplitudes in the resonant test, the present solution has been used to determine the dynamic Young's modulus of compacted clayey soil at high pressure subjected to longitudinal exciting forces in our experiments.

Formulation

Let us consider a compacted soil specimen in the form of a cylindrical rod excited by sinusoidal motion applied at the end. We may take the axis of the soil specimen as the Z-axis and use cylindrical coordinates r, θ and Z for difining the position of an element in the soil specimen. The components of displacements in the radial and tangential directions may be denoted U_r and U_θ and the component in the Z-direction by U_Z.

If we take the assumptions that the deformation of the specimen is symmetrical with respect to the Z-axis and the strain components in Z and r directions and the

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displacement in θ direction of an element in the soil specimen are represented by

$$\left.\begin{aligned} &\varepsilon_z(z,t)=\frac{\partial u_z(z,t)}{\partial z} \\ &\varepsilon_r(z,t)=\frac{\partial u_z(y,z,t)}{\partial r}=-\mu\varepsilon_z(z,t) \\ &u_\theta=0 \end{aligned}\right\} \qquad (1)$$

where ϵ_r, ϵ_z : strain components in r and Z directions
u_θ : displacement component in θ direction
μ : Poisson's ratio

the kinetic energy T per unit of length of the soil specimen is given by following equation

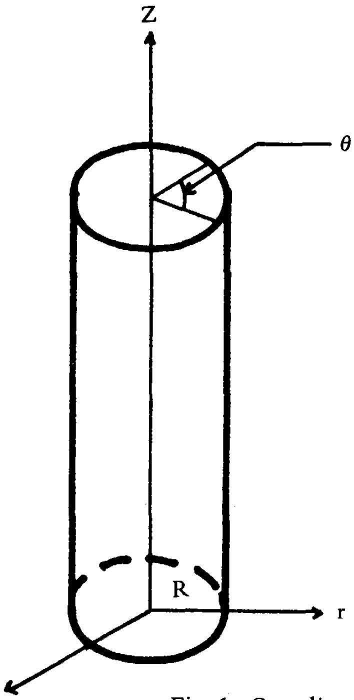

Fig. 1 Coordinates system of a soil specimen

$$T=\int_0^R\frac{\rho}{2}\left\{\left(\frac{\partial u_z}{\partial t}\right)^2+\left(\frac{\partial u_r}{\partial t}\right)^2\right\}2\pi r dr=\frac{1}{2}\rho\pi R^2\left(\frac{\partial u_z}{\partial t}\right)^2+\frac{1}{4}\rho\pi R^4\mu^2\left(\frac{\partial^2 u_z}{\partial t\partial z}\right)^2 \qquad (2)$$

where ρ : Mass density of soil specimen
R : radius of a cross section of the soil specimen
t : time

The strain energy per unit of length of the soil specimen W is

$$W=\int_0^R\left(\int_0^{\varepsilon z}E\varepsilon_z d\varepsilon_z\right)2\pi r dr=\frac{1}{2}\pi E R^2\left(\frac{\partial u_z}{\partial z}\right)^2 \qquad (3)$$

where E : Young's modulus of the soil specimen

Using Hamilton's principle, the variational equation of motion becomes

$$\delta\iint(T-W)\,dtdz=\iint\delta(T-W)\,dtdz=\delta\int dt\int\Big[\frac{1}{2}\rho\pi R^2\Big(\frac{\partial u_z}{\partial t}\Big)^2+\frac{1}{4}\rho\pi R^4\Big(\frac{\partial^2 u_z}{\partial t\partial z}\Big) -\frac{1}{2}\pi E R^2\Big(\frac{\partial u_z}{\partial z}\Big)^2\Big]dz \tag{4}$$

where the integration with respect to z is taken along the specimen.

Let T−W be replace temporarily by L, then we have

$$L=T-W=L\Big(\frac{\partial u_z}{\partial t},\frac{\partial u_z}{\partial z},\frac{\partial^2 u_z}{\partial t\partial z}\Big) \tag{5}$$

and first variation

$$\delta L=\frac{\partial L}{\partial\Big(\frac{\partial u_z}{\partial t}\Big)}\frac{\partial(\delta u_z)}{\partial t}+\frac{\partial L}{\partial\Big(\frac{\partial u_z}{\partial z}\Big)}\frac{\partial(\delta u_z)}{\partial z}+\frac{\partial L}{\partial\Big(\frac{\partial^2 u_z}{\partial t\partial z}\Big)}\frac{\partial^2(\delta u_z)}{\partial t\partial z}$$

$$=\rho\pi R^2\frac{\partial u_z}{\partial t}\frac{\partial(\delta u_z)}{\partial t}-E\pi R^2\frac{\partial u_z}{\partial z}\frac{\partial(\delta u_z)}{\partial z}+\frac{1}{2}\pi R^4\mu^2\rho\frac{\partial^2 u_z}{\partial t\partial z}\frac{\partial^2(\delta u_z)}{\partial t\partial z} \tag{6}$$

In forming the variation, we use the identities

$$\left.\begin{aligned}
&\frac{\partial u_z}{\partial t}\frac{\partial\delta u_z}{\partial t}+\frac{\partial^2 u_z}{\partial t^2}\delta u_z=\frac{\partial}{\partial t}\Big(\frac{\partial u_z}{\partial t}\delta u_z\Big)\\
&\frac{\partial u_z}{\partial z}\frac{\partial\delta u_z}{\partial z}+\frac{\partial^2 u_z}{\partial z^2}\delta u_z=\frac{\partial}{\partial z}\Big(\frac{\partial u_z}{\partial z}\delta u_z\Big)\\
&2\Big(\frac{\partial^2 u_z}{\partial z\partial t}\frac{\partial\delta^2 u_z}{\partial z\partial t}-\frac{\partial^4 u_z}{\partial z^2\partial t^2}\delta u_z\Big)\\
&=\frac{\partial}{\partial z}\Big(\frac{\partial^2 u_z}{\partial z\partial t}\frac{\partial^3 u_z}{\partial t}\delta u_z\Big)+\frac{\partial}{\partial t}\Big(\frac{\partial u_z}{\partial z\partial t}\frac{\partial\delta u_z}{\partial z}-\frac{\partial^3 u_z}{\partial z^2\partial t}\delta u_z\Big)
\end{aligned}\right\} \tag{7}$$

and on integrating by parts, and equating to zero the coefficient of δU_z under the sign of double integration:

$$\left.\begin{aligned}
&\iint \frac{\partial u_z}{\partial t}\frac{\partial(\delta u_z)}{\partial t}=\int\left(\frac{\partial u_z}{\partial t}\delta u_z-\int\frac{\partial^2 u_z}{\partial t^2}\delta u_z\,dt\right)dz=-\iint\frac{\partial^2 u_z}{\partial t^2}\delta u_z\,dt\,dz\\
&\iint \frac{\partial u_z}{\partial z}\frac{\partial(\delta u_z)}{\partial z}\,dz\,dt=\int\left(\frac{\partial u_z}{\partial z}\delta u_z-\int\frac{\partial^2 u_z}{\partial z^2}\delta u_z\,dz\right)dt=-\iint\frac{\partial^2 u_z}{\partial z^2}\delta u_z\,dz\,dt\\
&\iint \frac{\partial^2 u_z}{\partial t\partial z}\frac{\partial^2(\delta u_z)}{\partial t\partial z}\,dz\,dt=\int\left(\frac{\partial^2 u_z}{\partial t\partial z}\frac{\partial(\delta u_z)}{\partial t}-\int\frac{\partial^3 u_z}{\partial t\partial z^2}\frac{\partial(\delta u_z)}{\partial t}\,dz\right)dt\\
&=\left(\frac{\partial^2 u_z}{\partial t\partial z}\delta u_z-\int\frac{\partial^3 u_z}{\partial t^2\partial z}\delta u_z\,dt\right)-\int\left(\frac{\partial^3 u_z}{\partial t\partial z^2}\delta u_z-\int\frac{\partial^4 u_z}{\partial t^2\partial z^2}\delta u_z dt\right)dz\\
&=\iint\frac{\partial^4 u_z}{\partial t^2\partial z^2}\delta u_z\,dt\,dz
\end{aligned}\right\}\tag{8}$$

we obtain the equation

$$-\iint \pi R^2\left(-\rho\frac{\partial^2 u_z}{\partial t^2}+E\frac{\partial^2 u_z}{\partial z^2}+\frac{1}{2}\rho\mu^2 R^2\frac{\partial^4 u_z}{\partial t^2\partial z^2}\right)\delta u_z\,dt\,dz=0 \tag{9}$$

Consequently, we can reduce the following equation from the conditions which eq. (9) is satisfied for an arbitrary value of δU_z.

$$\rho\frac{\partial^2 u_z}{\partial t^2}=E\frac{\partial^2 u_z}{\partial z^2}+\frac{1}{2}\rho\mu^2 R^2\frac{\partial^4 u_z}{\partial t^2\partial z^2} \tag{10}$$

The eq. (10) is the expression for the longitudinal wave motion taking account of the inertia of the radial motion and the final term represents the correction of the radial inertia to conventional wave equation of thin rod.

Test Procedure

All vibration tests were carried out on cylindrical soil specimen 5cm in diameter and 10 – 10.5cm in length, compacted statically at high pressures 10 – 50 (kg/cm^2). The tests reported herein were performed on the diluvial clayey soils with the following index properties.

Table 1. Physical properties of Soils

Soil \ Property	Density	Liquid Limit	Plastic Limit	Plasticity Index	Opt. water content	Maximum dry desity
A	2.631	53.5%	23.3%	30.2	21.5%	1.609g/cm^3
B	2.664	46.5	30.4	16.1	20.0	1.680
C	2.661	82.2	53.4	28.8	49.3	1.094
D	2.771	70.9	47.1	23.8	30.5	1.325

Table 2. Mechanical analysis of sample soils

Soil \ Texture	Gravel	Sand	Silt	Clay	Triangular diagram
A	0%	32%	29%	39%	clay
B	0	3	87	10	silty loam
C	0	36	43	21	clay loam
D	0	24	44	32	clay

The compaction of soils was performed by special apparatus deviced for high compaction pressures in our laboratory and the conditions of soils – moisture content, density, degree of saturation and void ratio – were changed over a wide range.

Table 3. Conditions and sizes of a soil specimen

γ_d	Sr	A	ℓ	V
1.75 (g/cm^3)	0%	19.625 (cm^2)	10.5 (cm)	206.1 (cm^3)
	50			
	100			

An outline of the apparatus used in the longitudinal vibraiton tests is shown in Fig. 2. In general, this apparatus is called the resonant-column apparatus and the test is based on theoretical solutions of eq. (10) given in following section which relate the dynamic Young's modulus of the cylindrical soil specimen to its resonant frequency.

The soil specimen is attached to a vibrating base having a resonant frequency several times that of the soil specimen and accelerometer is attached to the top of the soil specimen. The electromagnetic vibration generator is used to apply the vibration to the base of the specimen. The frequency of the vibration generator may be adjusted to the maximum response of the soil specimen in order to determine its resonant frequency and the maximum amplitude of the soil specimen is measured by the synchro-scope connected the accelerometer on the top of the specimen.

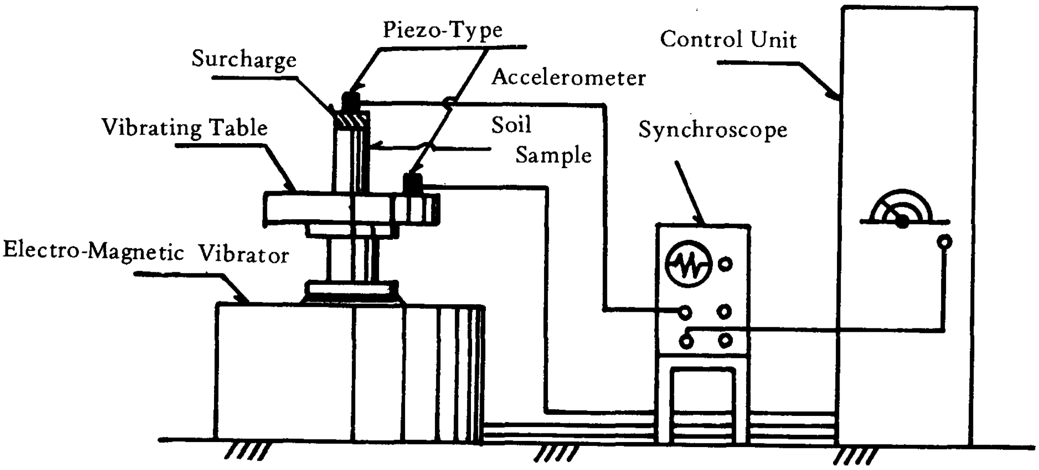

Fig. 2 Longitudinal Vibration Measuring Instrument

Solution and Boundary Conditions

When the vibration tests of soil specimens are carried out using the apparatus as shown in Fig. 2, the boundary conditions for the specimen may be written as follows:

$$Z=0 \quad : \quad u_z = u_0 \sin \omega t \tag{11}$$

$$Z=l \quad : \quad -\frac{W}{Ag}\left(\frac{\partial^2 u_z}{\partial t^2}\right) = E\varepsilon_z + \frac{1}{2}\rho\mu^2 R^2 \frac{\partial^2 \varepsilon_z}{\partial t^2}$$

$$= E\frac{\partial u_z}{\partial z} + \frac{1}{2}\rho\mu^2 R^2 \frac{\partial^3 u_z}{\partial t^2 \partial z} \tag{12}$$

where U_0 : amplitude of vibration table
ω : angular velocity (circular frequency) of vibration table
A : cross-section area of a soil specimen
W : weight of added mass and accelerometer
g : acceleration of gravity

The latter eq. (12) expresses the condition that the force of Z-direction in the specimen equals the product of the mass and its acceleration at the end with the mass attached. The sign is negative because the force is tensile or positive when the acceleration of the mass is negative.

As the motion of the specimen attached on the vibration table is stationary vibration, we may consider that the frequency of the specimen is equal to that of the table. Therefore, we can assume that the solution of eq. (10) is represented as follows:

$$u_z(z,t) = \lambda(z)\sin\,\omega t \tag{13}$$

where $\lambda(z)$: unknown function z only (the displacement amplitude along the length of the soil specimen)

This equation describes the displaced shape of a soil specimen vibrating in a natural mode.

Substituting eq. (13) into eq. (10), we have following equation

$$\left(\frac{E}{\rho} - \frac{1}{2}\rho\mu^2\ R^2\ \omega^2\right)\frac{d^2\lambda(z)}{dz^2} + \omega^2\,\lambda(z) = 0 \tag{14}$$

Introducing the notation

$$a^2 = \frac{E}{\rho} - \frac{1}{2}\rho u^2\ R^2\ \omega^2 \tag{15}$$

we can rewrite eq. (14) in the following form

$$\frac{d^2\lambda(z)}{dz^2} + \frac{\omega^2}{a^2}\lambda(z) = 0 \tag{16}$$

Consequently, the general solution of eq. (16) is given by the expression

$$\lambda(z) = C_1\cos\frac{\omega z}{a} + C_2\sin\frac{\omega z}{a} \tag{17}$$

where C_1 and C_2 are integral constants.

Substituting eq. (17) into eq. (13), we have the solution of the fundamental wave equation (10) as follows:

$$u_z(z,t) = \left(C_1\cos\frac{\omega z}{a} + C_2\sin\frac{\omega z}{a}\right)\sin\,\omega t \tag{18}$$

For a soil specimen of finite length, the displacement amplitude $\lambda(z)$ must be determined by the end conditions. Using the boundary conditions eq. (11) and eq. (12) to determine the integral constants C_1 and C_2 in eq. (17), we find

$$C_1 = U_0 \tag{19}$$

from the first boundary condition, and

$$C_2 = \left(\frac{1 + \frac{\rho g a A}{W} \tan\left(\frac{\omega \ell}{a}\right)}{\frac{\rho g a A}{W} \frac{1}{\omega} - \tan\left(\frac{\omega \ell}{a}\right)} \right) u_0 \tag{20}$$

from the second boundry condition.

When the weight of a soil specimen is represented by W', we have the following expression

$$W' = \rho g A \ell \tag{21}$$

So, we can rewrite eq. (19) as follows:

$$C_2 = \frac{u_0}{\tan(\phi - \beta)} \tag{22}$$

where $$\tan\phi = \frac{\alpha}{\beta}, \quad \frac{W'}{W} = \alpha \quad \text{and} \quad \frac{\omega \ell}{a} = \beta \tag{22'}$$

Substituting these values of C_1 and C_2 given by eq. (19) and eq. (22) into eq. (18), the displacement amplitude of the soil specimen will now be represented in the form

$$u_z(z,t) = u_0 \left(\cos \frac{\omega z}{a} + \frac{1}{\tan(\phi - \beta)} \sin \frac{\omega z}{a} \right) \sin \omega t \tag{23}$$

It is apparent that the displacement amplitude in eq. (23) $U_z(z,t)$ becomes infinite when we let $\tan(\phi - \beta)$ approch to zero, and the system of Fig. 3 occurs the phenomenon of resonance.

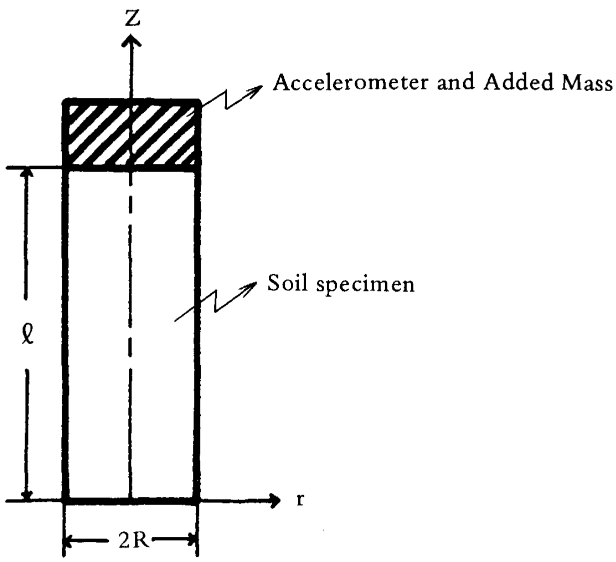

Fig. 3 Longditudinal Vibration System

Then,

$$\phi-\beta=(n-1)\pi \quad (n=1,2,3,........) \tag{24}$$

If a specimen can be excited longitudinally in the first normal mode of vibration and the resonant frequency f_R measured, we must take n = 1 in eq. (24).

Consequently,

$$\phi=\beta$$

and then

$$\tan^{-1}\left(\frac{\alpha}{\beta}\right)=\beta=\frac{\omega \ell}{a}$$

Finaly, we obtain the following relation

$$\frac{\omega \ell}{a}\tan\left(\frac{\omega \ell}{a}\right)=\beta \tan \beta=\frac{W'}{W} \tag{25}$$

Eq. (25) is the frequency equation for the first normal mode of longitudinal vibration. For any ratio of weight of soil specimen to weight of added mass, the values of β can be found. The minimum values of β which satisfy eq.(25) are plotted in Fig. 4 for various values of $\frac{W'}{W}$. The values of a for resonant frequency can now be computed from the following equation, using the appropriate values of β.

$$a=\frac{\omega \ell}{\beta}=\frac{2\pi f_R \ell}{\beta} \tag{26}$$

where $\omega=2\pi f_R$. $(\omega=2\pi f)$.

Consequently, dynamic Young's modulus E is founded from eq.(15) as follows:

$$E_{DR}=\left(\frac{2\pi f_R \ell}{\beta}\right)^2+2\rho\pi^2\mu^2R^2f_R^2=2\pi^2f_R^2\rho\left\{\frac{2\ell^2}{\beta^2}+\mu^2R^2\right\} \tag{27}$$

where f : frequency of vibration
f_R : resonant frequency

If we do not consider the effects of the radial inertia of cylindrical soil specimen, eq.(27) becomes

$$E_{DR1}=\left(\frac{2\pi f_R \ell}{\beta}\right)^2 \tag{28}$$

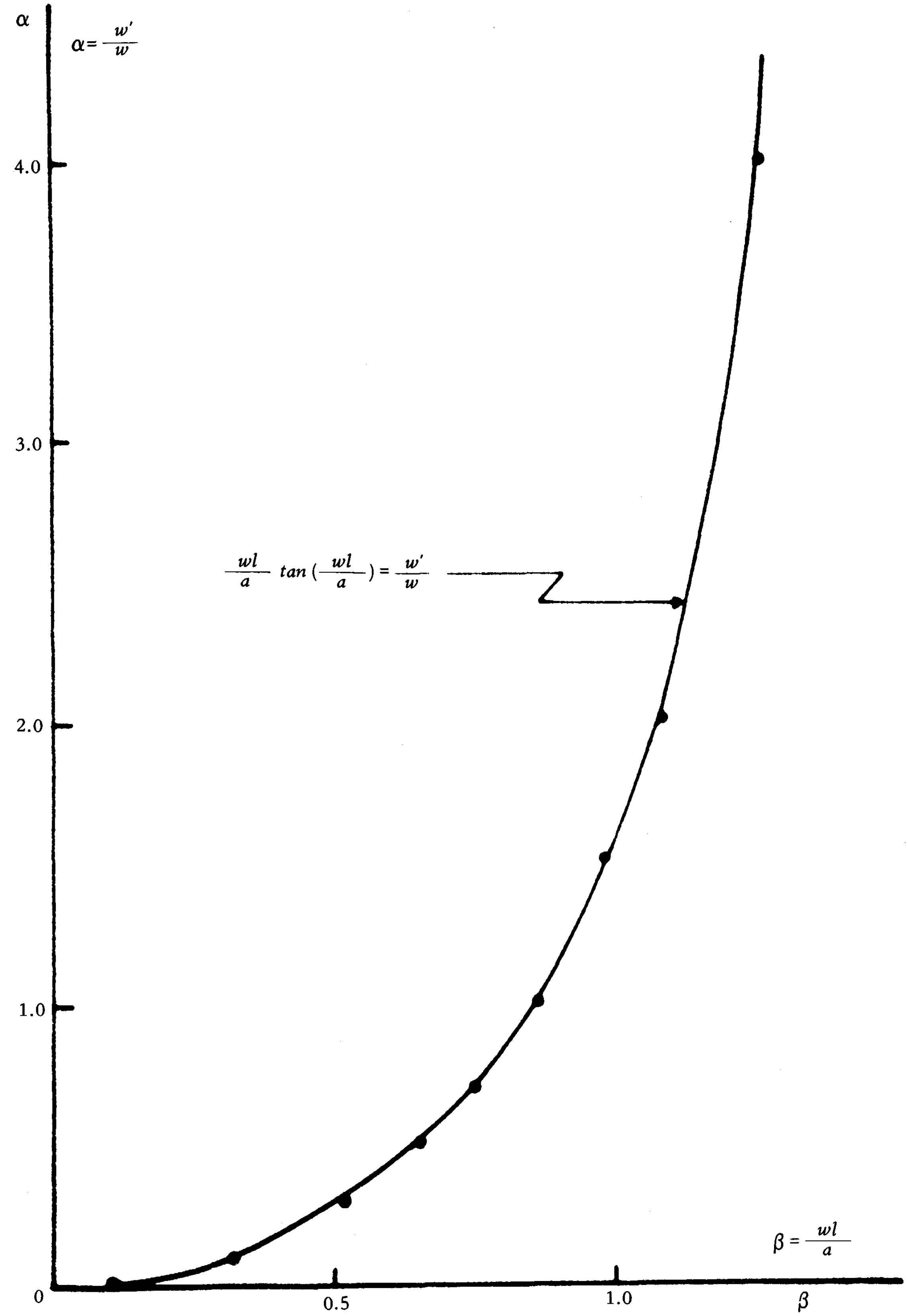

Fig. 4 Relationship between α and β

Test Results [4)]

The dynamic young's modulus of soil specimen under sinusoidal vibration was subjected to a constant loading in vertical direction. Measurements were taken of the acceleration in z-direction and the resonant frequency developed in each specimen.

The dynamic Young's modulus E was computed by eq.(27) from the data obtained in resonant tests, and these values were checked by the existing formula

$$E_{DR2} = 16f_R^2 \ell^2 \rho\left(1+\frac{2W}{\rho g A \ell}\right)+\frac{\pi \eta^2}{16\left(1+\frac{2W}{\rho g A l}\right)\rho l^2} \qquad (29)$$

where η is the coefficient of the viscosity of a soil.

As we may consider that the soil specimens show elastic behaviour in the range of small strain ($10^{-6} \sim 10^{-4}$), we take $\eta = 0$ in eq.(28).

Eq.(28) was introduced from the wave equation which based on the Voigt type rheological model of soil.

Fig. 5 ~ 8 shows the dynamic Young's moduli of soil A, B, C and D for the various values of $1/\alpha$. When we compare with the values of E which are computed by eq.(29) and eq.(28), the values by two different equations almost agree. The magnitude of E changes with degree of saturation or density, and a certain differences between $E_{Sr=0}$ and $E_{Sr=50}$ may be observed. When $0 < 1/\alpha < 2$, E shows small values in a measure, but this reason is not clear sufficiently according to the consideration done to date.

As shown in Fig. 9, the dynamic Young's modulus E increases with increasing

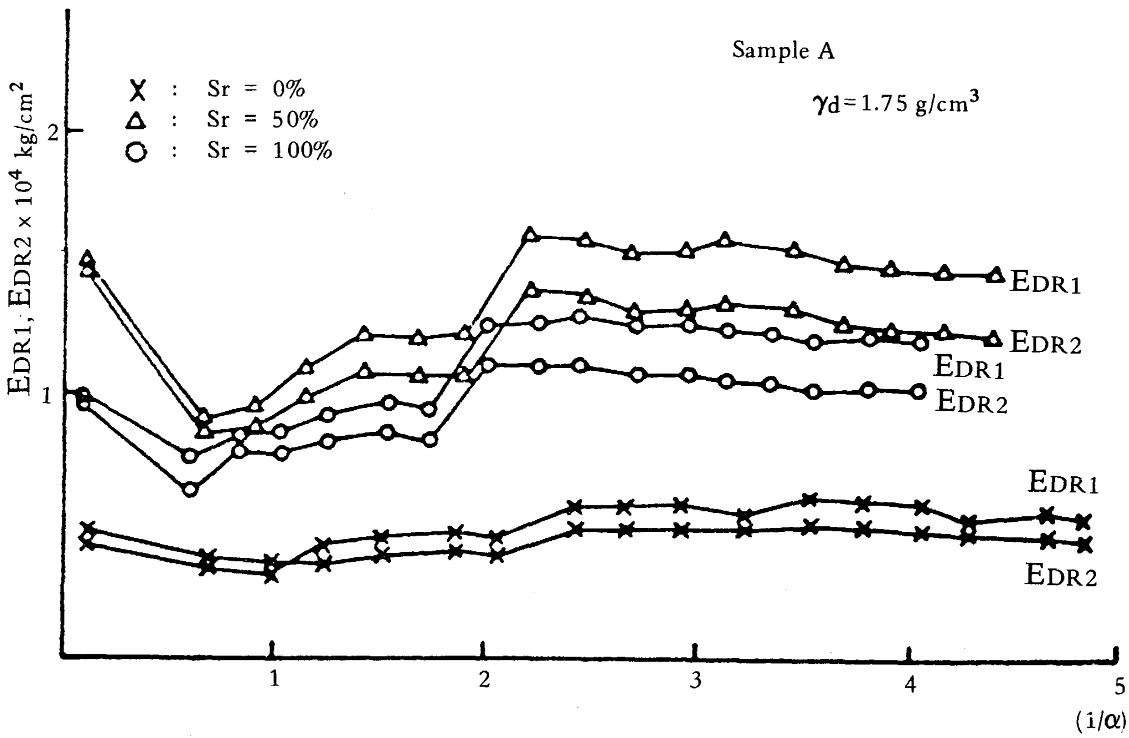

Fig. 5 Dynamic Young's Modulus (E_{DR1}, E_{DR2}) vs. Mass Ratio ($1/\alpha$)

moisture content to the neighbourhood of 10%, and then tend to return to the near original place. Also, as the value of density increase, the modulus E increase further.

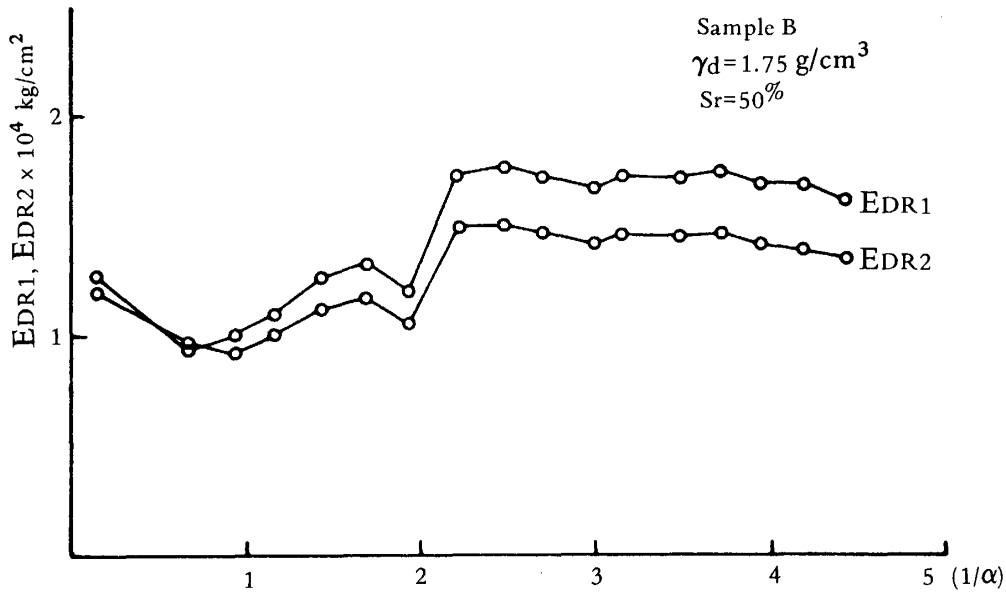

Fig. 6 Dynamic Young's Modulus (E_{DR1}, E_{DR2}) vs. Mass Ratio ($1/\alpha$)

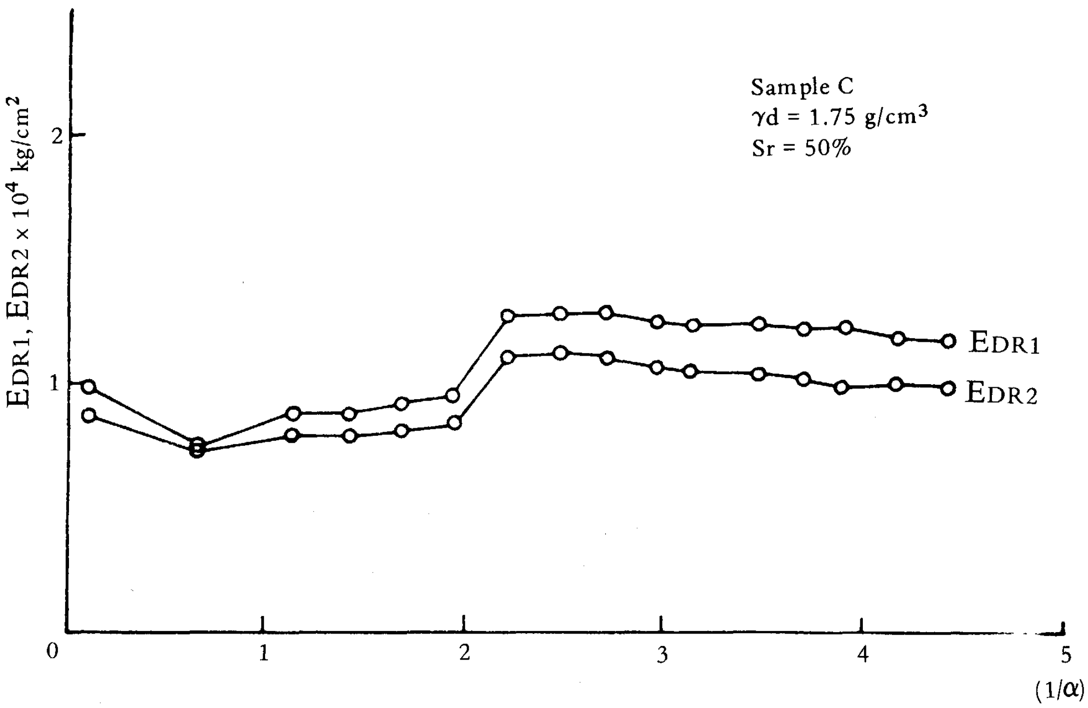

Fig. 7 Dynamic Youg's Modulus (E_{DR1}, E_{DR2}) vs. Mass Ratio ($1/\alpha$)

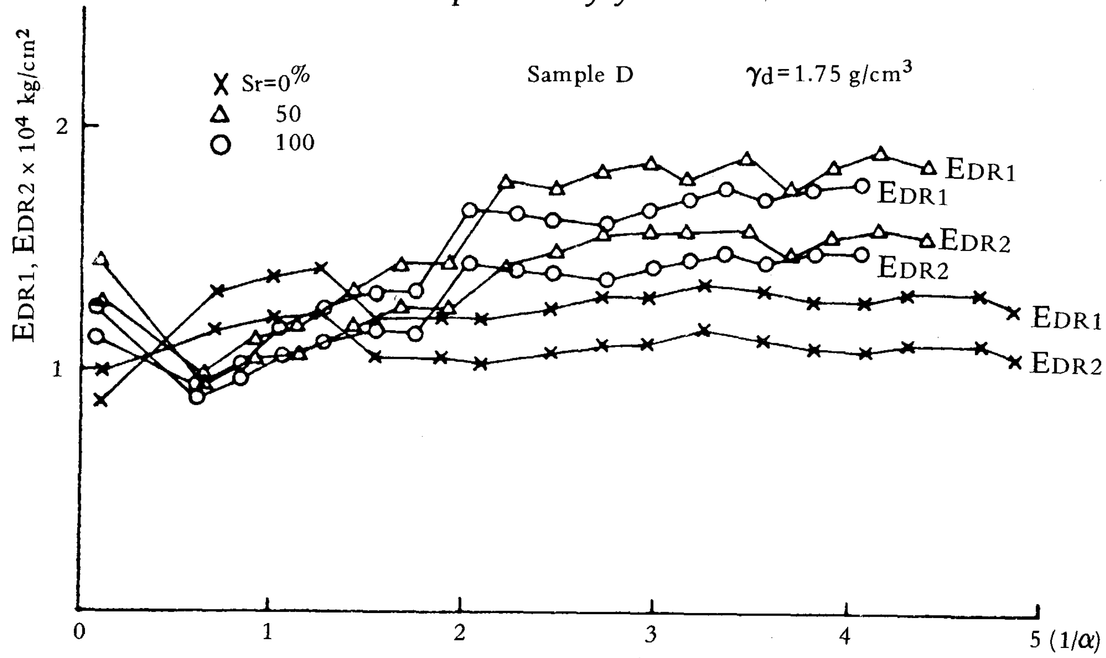

Fig. 8 Dynamic Young's Modulus (E_{DR1}, E_{DR2}) vs. Mass Ratio ($1/\alpha$)

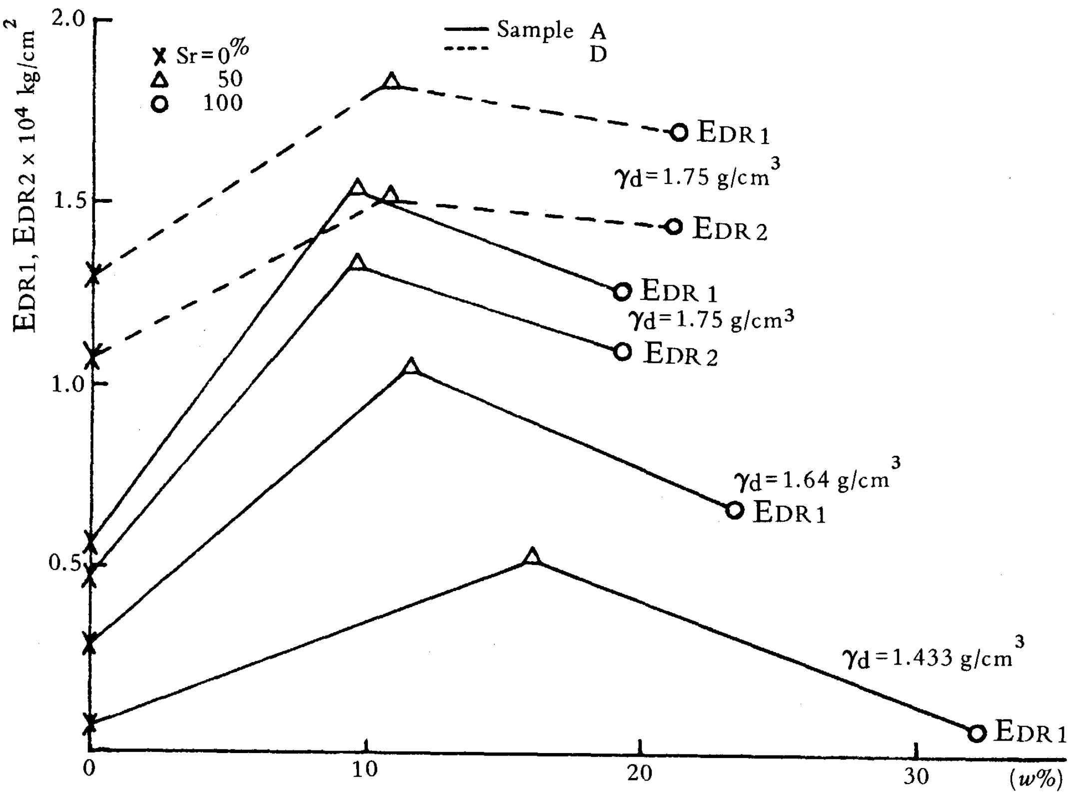

Fig. 9 Dynamic Young's Modulus (E_{DR1}, E_{DR2}) vs. Water Content (w)

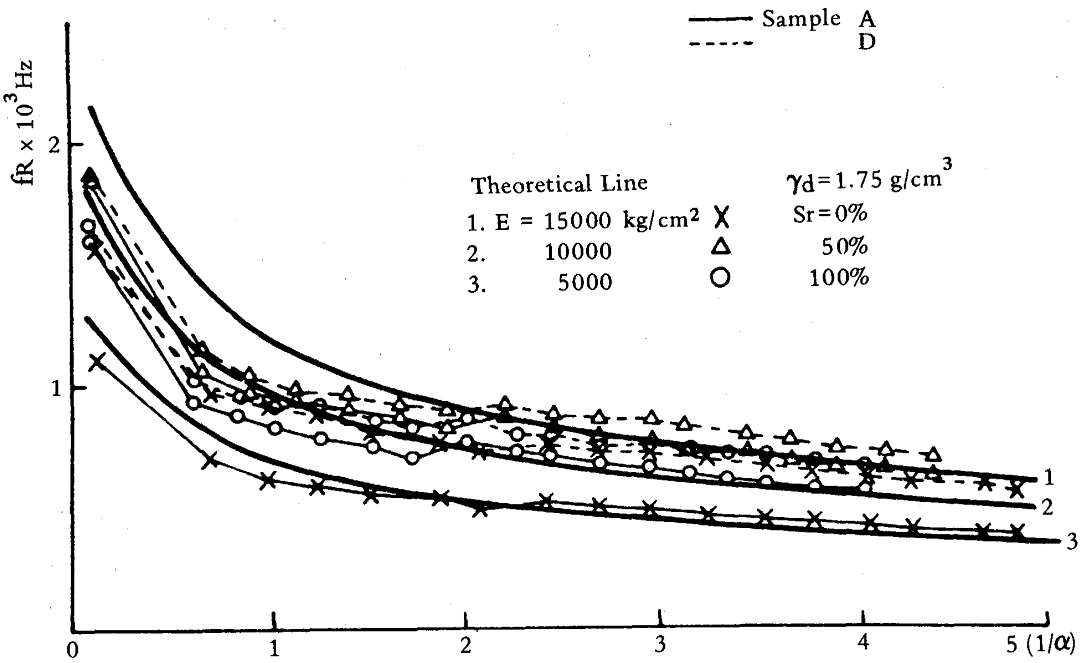

Fig. 10 Resonant Frequency (f_R) vs. Mass Ratio ($1/\alpha$)

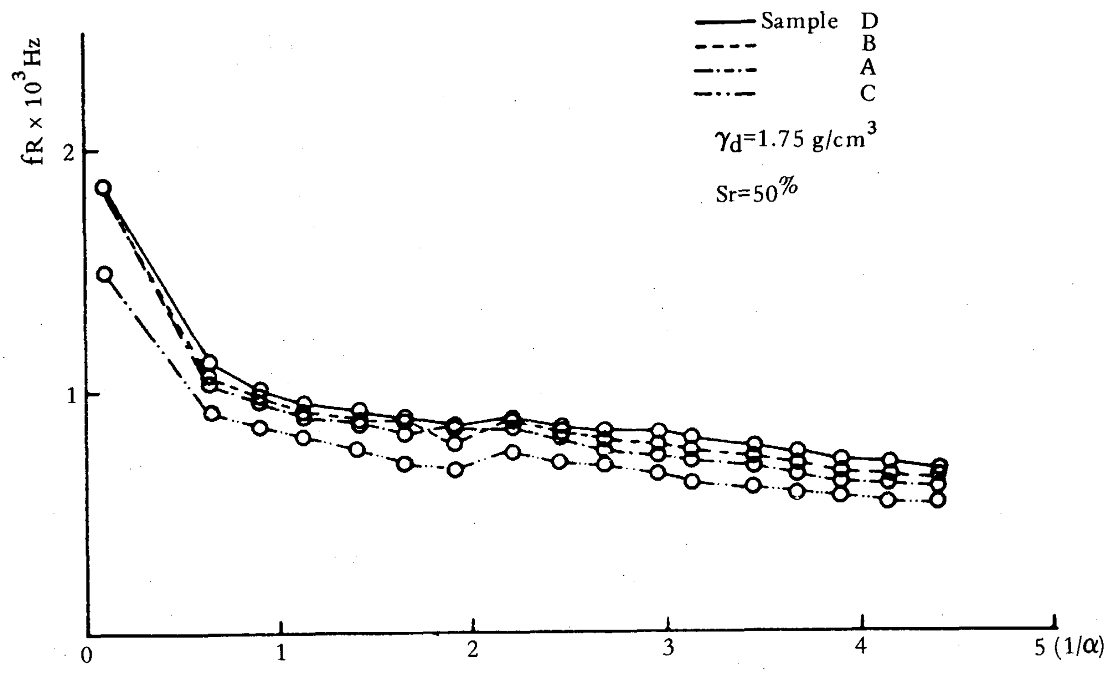

Fig. 11 Resonant Frequency (f_R) vs. Mass Ratio ($1/\alpha$)

Fig. 10 ~ 11 illustrates the relationships between the mass ratio $1/\alpha$ and the resonant frequency f_R of the soil specimen during which the specimens are the state of resonance. The resonant frequency decreases with increasingin $1/\alpha$, and $1/\alpha \sim f_R$ curves become a hyperbola and the curves approach to the $1/\alpha$ axis asymptotically. The experimental curves in Fig. 10 ~ 11 are calculated using the test data and these curves agree approximately with theoretical curves for E = 5000 kg/cm^2, E = 10000 kg/cm^2 and E = 15000 kg/cm^2. In the region of $1/\alpha = 0 \sim 1/\alpha = 2$, the values of the resonant frequency obtained from the tests are smaller than the theoretical results, but in the case of $S_r = 0$, the test results for any soils agree well with the theoretical values. Further, Fig. 10 ~ 11 illustrates the significant effect the attached mass has on the longitudinal wave velocity of compacted clayey soils. In these diagrams, we can find that the curves of resonant frequency show a rapid rate decrease in the region of $0 < 1/\alpha < 2$. For $1/\alpha > 2$, the curves become almost straight lines and each curves approach to constant values as $1/\alpha$ increases.

The effects of radial inertia on the dynamic Young's modulus were investigated by the term

$$2\rho\pi^2\mu^2 R^2 f_R^2 \tag{30}$$

Table 4. Effects of Lateral Inertia (P*) vs. Poisson's Ratio (μ) and Mass Ratio ($1/\alpha$)

$1/\alpha$ \ μ / P*	0.25	0.30	0.35	0.40	0.45
4.44	5.55	7.00	9.52	12.44	15.74
4.18	5.96	7.51	10.22	13.35	16.90
3.97	6.28	7.92	10.78	14.08	17.81
3.71	6.79	8.55	11.64	15.20	19.24
3.47	7.48	9.43	12.84	16.77	21.22
3.25	8.07	10.17	13.84	18.07	22.87
2.96	8.60	10.83	14.74	19.26	24.37
2.69	9.22	11.62	15.82	20.66	26.15
2.47	9.07	13.07	17.78	23.23	29.40
2.21	10.06	14.48	19.71	25.75	32.59
1.91	8.64	12.44	16.93	25.95	27.98
1.68	9.45	13.60	18.51	24.18	30.60
1.39	11.09	15.97	21.74	28.40	35.94
1.13	11.80	16.99	23.12	30.20	38.22
0.91	12.09	17.41	23.70	30.96	39.18
0.64	14.45	20.81	28.33	37.00	46.83
0.09	47.95	69.04	93.97	122.74	155.35

in eq.(27) using the experimental data. In this calculation, the results suggested that the radial inertia of soil specimen have not much effects upon the dynamic Young's modulus.

The effect of the dynamic Young's modulus due to radial inertia of a specimen is at most 1% in the value obtained from eq. (27), and so we can neglect the effects of radial inertia.

Conclusion

From the facts described above, we may conclude that eq.(28) is applicable and useful to find the dynamic Young's modulus of clayey soil by resonant coloumn test.

The analytical and experimental investigation indicate that the significant effects on dynamic Young's modulus arise when the additional mass is attached on the soil specimen.

From the analytical results, it can be consider that there is no effects of radial inertia on the dynamic Young's modulus.

Acknowledgements

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Table of Notation

A	:	Cross-section area of a soil specimen (cm^2)
E	:	Dynamic Young's Modulus (kg/cm^2)
E_{DR1}, E_{DR2}	:	Dynamic Young's Modulus (kg/cm^2) by eq. (29) and eq. (28)
f_R, f	:	Resonant Frequency (H_z), (c/s)
g	:	Acceleration of Gravity (cm/sec^2)
l	:	Length of Soil Specimen (cm)
P^*	:	Lateral Inertia = $2\pi^2\mu^2R^2f^2\rho$ (kg/cm^2)
$2R$	:	Diameter of a Compacted Soil Specimen (cm)
S_r	:	Degree of Saturation (%)
V	:	Volume of a Soil Specimen (cm^3)
w	:	Water Content (%)

α, $1/\alpha$	:	Mass Ratio
γ_d	:	Dry Density (g/cm^3)
μ	:	Dynamic Poisson's Ratio
ρ	:	Mass Density (g/cm^3)
ω	:	Circular Frequency (H_z), (c/s)