ON OPTIMAL CAPITAL INVESTMENT, RESEARCH AND DEVELOPMENT, AND ADVERTISING POLICIES

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1. Introduction

Problems of optimal investment have been studied by Arrow, Beckmann and Karlin [3], Arrow [1, 2], and others. Nerlove and Arrow [9] considered the problem of optimal advertising policy. Dhrymes [5] considered a model including both capital and advertising policies.

In the present paper we consider the problems of capital investment, research and development, and advertising policies, using the results obtained in optimal control theory. Many of the technical discussions that follow we owe to Lee and Markus [7].

2. Plant and Equipment Investment

2.1. Model

We begin by examining investments in physical plants and equipments. By physical plants and equipments we mean "capital stocks" that must be owned by the firm in order to employ their services in its production process. We consider such a firm as produces a single product by combining perfectly variable factors of production with physical plants and equipments or capital stocks. By a perfectly variable factor is meant one that can be altered in amount according to a cost schedule which is independent of either the time rate of change in the amount of that factor used or the time interval between a decision to vary the amount of that factor and its actual variation.

Let $x_1, x_2, \cdots, x_n$ denote capital stocks, and define a vector x as

$$x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

For simplicity we assume constant rates of utilization of the services of capital stocks. Furthermore we assume that the amounts of the perfectly variable factors used with physical plants and equipments of given sizes are always optimally adjusted to the production technology and the prices of the perfectly variable factors. In addition, we assume that output of the product is determined to maximize the difference between the revenue obtained by sales of the product and the production costs. We call this difference "operating profit", and let Π denote the operating profit function.

According to the above arguments the rate of operating profit may be regarded as a function of capital stocks, production technology and prices of the perfectly variable factors. In this section we do not introduce research and development policy of the firm, so we assume that no technological progress takes place. Therefore the rate of operating profit may be denoted by $\Pi(x, t)$. The variable t included in the function Π signifies the incidences of changes in prices of perfectly variable factors.

Next we introduce an expansion cost (investment outlay) function. Let I_i be the rate of gross investment in i-th capital good, and define a vector I as

$$I=\begin{pmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{pmatrix}.$$

At each moment of time $S(I, t)$ denotes the expansion cost. This says that the cost of expansion depends both on the rates of gross investments and on the time that has elapsed between the beginning of the planning period of the firm, date 0, and the date at which the investments occur, date t. The variable t included in the function S signifies the incidence of changes in supply conditions of capital goods.

Then the net receipt at time t, say $R(t)$, is given by

$$R(t)=\Pi[x(t), t]-S[I(t), t].$$

Present value is defined as the integral of discounted net receipts:

$$\int_0^T e^{-\int_0^t r(\tau)d\tau} R(t)dt,$$

where $r(\tau)$ is the rate of time discount at time τ and T is the length of the planning period. For simplicity of the following discussion, we assume that the rate of time discount is a constant r, and that the length of the planning period is a fixed finite positive number. Accordingly, the present value defined above may be represented in a simpler form:

$$\int_0^T e^{-rt} R(t)dt. \tag{2.1}$$

The aim of the firm is to maximize the present value (2.1), or to minimize the functional

$$J(I)=-\int_0^T e^{-rt}R(t)dt=\int_0^T \{S[I(t),\ t]-\Pi[x(t),\ t]\}e^{-rt}dt \tag{2.2}$$

by suitable choice of investment policy $I(t)$.

The evolution of capital stocks is determined by their initial states and by the investment policy chosen. If we assume depreciation at fixed exponential rates δ_i $(i=1,\ 2,\ \cdots,\ n)$, then

$$\dot{x}_i=I_i(t)-\delta_i x_i(t)$$

or in a matrix form

$$\dot{x}=I(t)-\delta x(t), \tag{2.3}$$

where the dot denotes differentiation with respect to time and δ is defined as

$$\delta=\begin{pmatrix} \delta_1 & 0 & \cdots & 0 \\ 0 & \delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_n \end{pmatrix}.$$

Let x_0 be the initial state vector of capital stocks. Then the initial condition is given by

$$x(0)=x_0, \tag{2.4}$$

where the vector x_0 is defined as

$$x_0=\begin{pmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{pmatrix}.$$

Now we state assumptions that are held in sections 2.2.1. and 2.2.2.

(1) $I(t)$ is a Lesbegue measurable vector-valued function on a finite interval $0\leqq t\leqq T$ and its values must satisfy the constraint

$$0\leqq I(t)\leqq I_{\max} \qquad (0\leqq t\leqq T), \tag{2.5}$$

where $I_{\max}$ is defined by

$$I_{\max}=\begin{pmatrix} I_{1\max} \\ I_{2\max} \\ \vdots \\ I_{n\max} \end{pmatrix},\ 0<I_{i\max}<+\infty \qquad (i=1,\ 2,\ \cdots,\ n).$$

Let $\Omega\subset R^n$ denote the set of all I that satisfy the constraint (2.5), where R^n is the n-dimensional vector space.

Then, for each choice of the function $I(t) \subset \Omega$, the differential system (2.3) has a unique absolutely continuous solution $x(t)$ on a subinterval of $[0, T]$, with a prescribed initial state $x_0 = x(0)$. This is the conclusion of the Carathéodory existence theorem for differential systems.(1) The solution $x(t)$ will be called the corresponding trajectory to the investment policy $I(t)$. The trajectory $x(t)$ is continuous, and it has a derivative, except on a set of measure zero, such that the differential system (2.3) is satisfied almost everywhere.

(2) The operating profit function Π is concave and differentiable with respect to x, and $\Pi \geqq 0$ and $\partial \Pi / \partial x > 0$ for every $x > 0$.

(3) The expansion cost function S is strictly convex and differentiable with respect to I, and $S \geqq 0$ and $\partial S / \partial I > 0$ for every $I > 0$.(2)

For the purpose of the following discussion we need some definitions.(3)

(i) A target set. In the cases in which values of the trajectory of capital stocks $x(t)$ are required to be in some prescribed set in the x-space R^n, we call the set a target set.

(ii) An admissible investment policy. An investment policy $I(t) \subset \Omega$ which steers the initial state of capital stocks x_0 to the prescribed target set is called an admissible investment policy. Let Δ denote the class of admissible investment policies.

(iii) An optimal investment policy. An investment policy $I^*(t)$ in Δ is called optimal in case

$$J(I^*) \leqq J(I)$$

for all $I(t)$ in Δ.

(iv) A controllable process. The differential system (2.3) is called a control process in optimal control theory. A control process is called a controllable process if there exists at least one admissible investment policy.

(v) A set of attainability. Consider the control process (2.3) with the initial state of capital stocks x_0, and investment policies $I(t) \subset \Omega$ on $0 \leqq t \leqq T$. Let $x(t)$ denote the corresponding trajectory of capital stocks initiating at $x(0) = x_0$. The set of attainability denoted by $A(T)$ is the set of all endpoints $x(T)$ in R^n.

(vi) An extremal investment policy. An investment policy $I(t) \subset \Omega$ on $0 \leqq t \leqq T$ which steers x_0 to an endpoint $x(T)$ in the boundary $\partial A(T)$ of the set of attainability by the process (2.3) is called extremal, and the

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(1) cf. Coddington and Levinson [4], Lee and Markus [6, 7].

(2) This author treated in the paper [8] the case in which the expansion cost is given by the product of capital good prices and the rates of gross investments.

(3) Some additional concepts will be defined later.

corresponding trajectory $x(t)$ is an extremal trajectory of capital stocks.

(vii) A normal process. A control process is called normal in case any two investment policies $I^1(t)$ and $I^2(t) \subset \Omega$ on $0 \leqq t \leqq T$, which steer x_0 to the same boundary point in $A(T)$, must be equal almost everywhere on $0 \leqq t \leqq T$.

2.2. Optimal Investment Policy

2.2.1. In the first instance we consider the case in which no requirement is imposed on the values of capital stocks over the planning period $0 \leqq t \leqq T$, that is, the target set is the x-space R^n itself over the planning period.(4)

Then our problem is formulated as follows.

Among all investment policies that steer the initial state of capital stocks x_0 to some point x in R^n according to the differential system

$$\dot{x} = I(t) - \delta x(t) \tag{2.3}$$

and satisfy the constraint

$$0 \leqq I(t) \leqq I_{\max} \qquad (0 \leqq t \leqq T), \tag{2.5}$$

find one for which the objective functional

$$J(I) = \int_0^T \{S[I(t),\ t] - \Pi[x(t),\ t]\} e^{-rt} dt \tag{2.2}$$

achieves the least possible value.

Now let us consider the set of attainability $A(T) \subset R^n$ of all endpoints $x(T)$ of the trajectory of capital stocks $x(t)$ initiating at $x(0) = x_0$. Making reference to the well known result in optimal control theory,(5) we can prove the following statement.

Proposition 1. Consider the control process relating the state of capital stocks $x(t)$ to the investment policy $I(t)$ on $0 \leqq t \leqq T$

$$\dot{x} = I(t) - \delta x(t) \tag{2.3}$$

with intial state x_0 and constraint on the investment policy

$$0 \leqq I(t) \leqq I_{\max} \qquad (0 \leqq t \leqq T). \tag{2.5}$$

Then the set of attainability $A(T)$ is compact and convex in R^n.

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(4) In this case it is selfevident that the process (2.3) initiating at $x(0) = x_0$ is controllable. The variation-of-parameter formula states that the solution of the differential system (2.3) with initial condition (2.4) is

$$x(t) = e^{-t\delta} x_0 + \int_0^t e^{-(t-\tau)\delta} I(\tau) d\tau,$$

so $x(t) \geqq 0$ for investment policies $I(t) \subset \Omega$ on $0 \leqq t \leqq T$.

(5) cf. Schmaedeke [10], or Lee and Markus [7].

Proof. To prove that $A(T)$ is compact, we prove that every sequence of points $\{x^{\nu}(T)\}$ ($\nu=1, 2, \cdots$) in $A(T)$ has a subsequence which converges to some limit point $\bar{x}(T)\in A(T)$. Consider the solution $x^{\nu}(t)$ of the differential system (2.3) corresponding to $x^{\nu}(T)$, and the investment policy $I^{\nu}(t)\subset\Omega$ on $0\leqq t\leqq T$ which yields the trajectory $x^{\nu}(t)$. Then the variation-of-parameter formula states

$$x^{\nu}(t)=e^{-t\delta}x_0+\int_0^t e^{-(t-\tau)\delta}I^{\nu}(\tau)d\tau. \tag{2.6}$$

As is well known, the family of all measurable vector functions on a prescribed compact real interval each of whch has values in a prescribed compact convex set is sequentially weakly compact.(6) Therefore the set of all investment policies $I(t)\subset\Omega$ on $0\leqq t\leqq T$ is weakly compact, and there is a subsequence $\{I^{\nu_i}(t)\}$ ($i=1, 2, \cdots$) which converges weakly to an investment policy $\bar{I}(t)\subset\Omega$ on $0\leqq t\leqq T$, that is,

$$\lim_{i\to\infty}\int_0^t e^{-(t-\tau)\delta}I^{\nu_i}(\tau)d\tau=\int_0^t e^{-(t-\tau)\delta}\bar{I}(\tau)d\tau. \tag{2.7}$$

Let $\bar{x}(t)$ denote the corresponding trajectory to $\bar{I}(t)$. Then from (2.6) and (2.7)

$$\bar{x}(t)=e^{-t\delta}x_0+\int_0^t e^{-(t-\tau)\delta}\bar{I}(\tau)d\tau=\lim_{i\to\infty}x^{\nu_i}(t),$$

therefore

$$\lim_{i\to\infty}x^{\nu_i}(T)=\bar{x}(T)\in A(T).$$

Thus the set $A(T)$ is compact.

To prove that $A(T)$ is convex we prove that the line segment

$$(1-\mu)x^1(T)+\mu x^2(T) \qquad (0\leqq\mu\leqq 1)$$

joining any two points $x^1(T)$ and $x^2(T)$ in $A(T)$, lies entirely in $A(T)$. Let $x^1(t)$ and $x^2(t)$ be the solutions of the differential system (2.3) corresponding to investment policies $I^1(t)$ and $I^2(t)\subset\Omega$ on $0\leqq t\leqq T$ respectively. And define an investment policy $I^{\mu}(t)\subset\Omega$ on $0\leqq t\leqq T$ by

$$I^{\mu}(t)=(1-\mu)I^1(t)+\mu I^2(t).$$

Then the corresponding trajectory of capital stocks $x^{\mu}(t)$ is given by

$$x^{\mu}(t)=(1-\mu)[e^{-t\delta}x_0+\int_0^t e^{-(t-\tau)\delta}I^1(\tau)d\tau]$$
$$+\mu[e^{-t\delta}x_0+\int_0^t e^{-(t-\tau)\delta}I^2(\tau)d\tau].$$

(6) cf. Lee and Markus [7], pp. 157–160.

Therefore we have

$$x^{\mu}(t)=(1-\mu)x^{1}(t)+\mu x^{2}(t)$$
$$x^{\mu}(T)=(1-\mu)x^{1}(T)+\mu x^{2}(T).$$

Thus $A(T)$ is convex. (Q.E.D.)

Next we examine investment policies which steer x_0 to a point in $\partial A(T)$. Such an investment policy is called extremal. In order to express the extremality conditions we consider the differential system

$$\dot{x}=-\delta x(t)$$

and its adjoint system

$$\dot{\eta}=\eta(t)\delta,$$

where $\eta(t)$ is an n-row vector. This adjoint system has a solution of the form $\eta(t)=\eta_0 e^{t\delta}$, where η_0 is a constant vector. If $\eta_0 \neq 0$, then $\eta(t)$ is a nontrivial solution of the adjoint system.

Proposition 2. Consider the control process relating the state of capital sotcks $x(t)$ to the investment policy $I(t)$ on $0 \leqq t \leqq T$

$$\dot{x}=I(t)-\delta x(t) \tag{2.3}$$

with intial state x_0 and constraint on the investment policy

$$0 \leqq I(t) \leqq I_{\max} \qquad (0 \leqq t \leqq T). \tag{2.5}$$

An investment policy $I(t) \subset \Omega$ on $0 \leqq t \leqq T$ is extremal if and only if there exists a nontrivial solution $\eta(t)$ of the adjoint system

$$\dot{\eta}=\eta(t)\delta \tag{2.8}$$

such that

$$\eta(t)I(t)=\max_{I \in \Omega} \eta(t)I \tag{2.9}$$

for almost all t on $0 \leqq t \leqq T$.

Proof. First, assuming the investment policy $I(t)$ on $0 \leqq t \leqq T$ to be extremal, we prove the existence of such a solution of the adjoint system that is required by the *Proposition*.

Let $I(t)$ be an extremal investment policy which steers x_0 to $x(T) \in \partial A(T)$ by the corresponding trajectory

$$x(t)=e^{-t\delta}x_0+\int_0^t e^{-(t-\tau)\delta}I(\tau)d\tau.$$

Since the set of attainability $A(T)$ is compact and convex, $A(T)$ has a supporting hyperplane at the boundary point $x(T)$. Let $\eta(T)$ be the outward unit normal vector to the hyperplane at $x(T)$. And define the nontrivial adjoint trajectory $\eta(t)$ by

$$\eta(t)=\eta_0 e^{t\delta} \text{ with } \eta(T)=\eta_0 e^{T\delta}.$$

Then we have

$$\eta(t)x(t)=\eta_0 x_0+\int_0^t \eta_0 e^{\tau\delta} I(\tau)d\tau$$
$$=\eta_0 x_0+\int_0^t \eta(\tau)I(\tau)d\tau,$$

so

$$\eta(T)x(T)=\eta_0 x_0+\int_0^T \eta(\tau)I(\tau)d\tau. \tag{2.10}$$

Now suppose that the maximum condition (2.9) is not satisfied on some set of positive duration in $0\leqq t\leqq T$, that is,

$$\eta(t)I(t)<\max_{I\in\Omega}\eta(t)I,$$

and define an investment policy $\tilde{I}(t)\subset\Omega$ on $0\leqq t\leqq T$ by

$$\eta(t)\tilde{I}(t)=\max_{I\in\Omega}\eta(t)I.$$

Let $\tilde{x}(t)$ denote the corresponding trajectory to the investment policy $\tilde{I}(t)$. Then we have

$$\eta(T)\tilde{x}(T)=\eta_0 x_0+\int_0^T \eta(\tau)\tilde{I}(\tau)d\tau. \tag{2.11}$$

By the definition of the investment policy $\tilde{I}(t)$ we have

$$\int_0^T \eta(\tau)I(\tau)d\tau<\int_0^T \eta(\tau)\tilde{I}(\tau)d\tau,$$

therefore from (2.10) and (2.11) we have

$$\eta(T)x(T)<\eta(T)\tilde{x}(T). \tag{2.12}$$

However the inequality (2.12) contradicts the construction of $\eta(T)$ as the outward normal vector at the boundary point $x(T)$. The inequality (2.12) implies that the point $x(T)\in\partial A(T)$ is separated by the supporting hyperplane from the set of attainability $A(T)$. But this is impossible because the point $x(T)$ belongs to the boundary $\partial A(T)$.

Thus we can conclude that

$$\eta(t)I(t)=\max_{I\in\Omega}\eta(t)I$$

for almost all t on $0 \leqq t \leqq T$.

Conversely we assume that, for some nontrivial adjoint trajectory $\eta(t)=\eta_0 e^{t\delta}$, the investment policy $I(t)\subset\Omega$ on $0\leqq t\leqq T$ satisfies the maximum condition (2.9) almost everywhere on $0\leqq t\leqq T$, and prove that the investment policy is extremal, that is, the corresponing trajectory of capital stocks $x(t)$ terminates at a boundary point of $A(T)$.

Now suppose that the terminal state $x(T)$ lies in the interior of $A(T)$, and consider a point $\tilde{\tilde{x}}(T)$ in $A(T)$ such that

$$\eta(T)x(T)<\eta(T)\tilde{\tilde{x}}(T)$$

for the specified adjoint trajectory $\eta(t)$. Let $\tilde{\tilde{I}}(t)\subset\Omega$ on $0\leqq t\leqq T$ denote the investment policy which yields the trajectory $\tilde{\tilde{x}}(t)$. Then by the above supposition we have

$$\eta(t)\tilde{\tilde{I}}(t)\leqq\eta(t)I(t)=\max_{I\in\Omega}\eta(t)I$$

for almost all t on $0\leqq t\leqq T$. Computing the inner products $\eta(t)x(T)$ and $\eta(T)\tilde{\tilde{x}}(T)$ we obtain

$$\eta(T)x(T)\leqq\eta(T)\tilde{\tilde{x}}(T).$$

But this is a contradiction. Therefore the terminal state of capital stocks denoted by $x(T)$ must lie in the boundart $\partial A(T)$. (Q.E.D.)

Now we procced to prove that the control process (2.3) is normal, that is, any two investment policies which steer x_0 to the same boundary point of $A(T)$ by the process (2.3) must be equal almost everywhere in $0\leqq t\leqq T$.

Proposition 3. Consider the control process relating the state of capital stocks $x(t)$ to the investment policy $I(t)$ on $0\leqq t\leqq T$

$$\dot{x}=I(t)-\delta x(t) \tag{2.3}$$

with initial state x_0 and constraint on the investment policy

$$0\leqq I(t)\leqq I_{\max} \qquad (0\leqq t\leqq T). \tag{2.5}$$

Then any two investment policies with values in $\Omega\subset R^n$ on $0\leqq t\leqq T$ are equal for almost all t on $0\leqq t\leqq T$.

Proof. Let $\bar{I}(t)\subset\Omega$ on $0\leqq t\leqq T$ be an extremal investment policy that steers the initial state of capital stocks x_0 to the terminal state $\bar{x}(T)$ in $\partial A(T)$, and $\bar{x}(t)$ be the corresponding trajectory of capital stocks.

Since the set of attainability $A(T)$ is compact and convex (*Proposition 1*), $A(T)$ has a supporting hyperplane at the boundard point $\bar{x}(T)$. Let $\eta(t)$ be a nontrivial adjoint trajectory with $\eta(T)$ an outward normal vector of $A(T)$ at $\bar{x}(T)$. Then we obtain the maximum condition

$$\eta(t)\bar{I}(t)=\max_{I\in\Omega}\eta(t)I$$

almost everywhere on $0\leq t\leq T$.

Now let $I(t)\subset\Omega$ on $0\leq t\leq T$ denote an investment policy which steers x_0 to the same terminal state $\bar{x}(T)$ in $\partial A(T)$, and let $x(t)$ be the corresponding trajectory. In the following discussion we show that $\bar{I}(t)=I(t)$ almost everywhere on $0\leq t\leq T$.

Suppose that, in some subinterval of $[0, T]$ with positive duration,

$$\eta(t)I(t)<\max_{I\in\Omega}\eta(t)I.$$

By using the variation-of-parameter formula, compute $x(T)$ and $\eta(t)$, then we obtain the inner product

$$\eta(T)x(T)=\eta_0 x_0+\int_0^T \eta(t)I(t)dt.$$

Now specialize $I(t)$ to $\bar{I}(t)$ and $x(t)$ to $\bar{x}(t)$ to obtain

$$\eta(T)\bar{x}(T)=\eta_0 x_0+\int_0^T \eta(t)\bar{I}(t)dt.$$

Therefore, under the above supposition, we obtain

$$\eta(T)x(T)<\eta(T)\bar{x}(T).$$

But this inequality contradicts to the assumption that the trajectory of capital stocks $x(t)$ terminates at the point $\bar{x}(T)$. Thus the investment policy $I(t)\subset\Omega$ on $0\leq t\leq T$ which steers x_0 to the same terminal state $\bar{x}(T)$ in $\partial A(T)$ must statisfy the maximum condition

$$\eta(t)I(t)=\eta(t)\bar{I}(t)=\max_{I\in\Omega}\eta(t)I \tag{2.13}$$

almost everywhere on $0\leq t\leq T$.

Furthermore, to satisfy the condition (2.13) for the same $\eta(t)$, $\bar{I}(t)$ and $I(t)$ must be equal almost everywhere on the planning period. Thus the control process (2.3) is normal. (Q.E.D.)

Proposition 3 says that the investment policy on $0\leq t\leq T$ which steers x_0 to each endpoint in $\partial A(T)$ is unique. Using the result that the process (2.3) is normal we can strengthen the result of *Proposition 1*.

Proposition 4. Consider the control process relating the state of capital stocks $x(t)$ to the investment policy $I(t)$ on $0\leq t\leq T$

$$\dot{x}=I(t)-\delta x(t) \tag{2.3}$$

with initial state x_0 and constraint on the investment policy

$$0 \leqq I(t) \leqq I_{\max} \qquad (0 \leqq t \leqq T). \tag{2.5}$$

Then the set of attainability $A(T)$ is strictly convex.

Proof. Conversely assume that $A(T)$ is not strictly convex. Since $A(T)$ is compact and convex as proved in *Proposition 1*, there is a supporting hyperplane p. Suppose that the intersection $p \cap A(T)$ contains more than one point. Then there is a compact line segment l in $p \cap A(T)$ that combines these points. Let $I^a(t)$ and $I^b(t) \subset \Omega$ on $0 \leqq t \leqq T$ denote two investment policies that steer x_0 to distinct points P^a and P^b in l respectively.

Now let $\mathscr{T}$ denote the interval $0 \leqq t \leqq T$. Consider, for each measurable subinterval $B \subset \mathscr{T}$, a real $2n$-vector

$$\Gamma(B) = \begin{pmatrix} \int_B e^{\tau\delta} I^a(\tau) d\tau \\ \int_B e^{\tau\delta} I^b(\tau) d\tau \end{pmatrix}.$$

Then the vector-valued set function $\Gamma(B)$ has some values

$$\Gamma(B) = \begin{bmatrix} \beta^a \\ \beta_b \end{bmatrix} \text{ and } \Gamma(\phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where ϕ is an empty set. According to Liapnov's result on measure theory,[7] there exists a subset $B_{\frac{1}{2}}$ for which[8]

$$\Gamma(B_{\frac{1}{2}}) = \Gamma(\mathscr{T} - B_{\frac{1}{2}}) = \begin{pmatrix} \beta_{\frac{a}{2}} \\ \beta_{\frac{b}{2}} \end{pmatrix}. \tag{2.14}$$

Since P^a and P^b are distinct, β is not equal to β and so neither $B_{\frac{1}{2}}$ and $(\mathscr{T} - B_{\frac{1}{2}})$ is a null set.

Define the investment policies $I^1(t)$ and $I^2(t)$ as follows:

$$I^1(t) = \begin{cases} I^a(t) & \text{for } t \in B_{\frac{1}{2}} \\ I^b(t) & \text{for } t \in (\mathscr{T} - B_{\frac{1}{2}}) \end{cases}$$

and

$$I^2(t) = \begin{cases} I^a(t) & \text{for } t \in (\mathscr{T} - B_{\frac{1}{2}}) \\ I^b(t) & \text{for } t \in B_{\frac{1}{2}} \end{cases}$$

Then the terminal states of the corresponding trajectries $x^1(t)$ and $x^2(t)$ are respectively given by

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(7) cf. Lee and Markus [7], pp. 163–164.
(8) $(\mathscr{T} - B_{\frac{1}{2}})$ is a difference set.

$$\left.\begin{aligned} x^1(T)&=e^{-T\delta}x_0+e^{-T\delta}\int_{B_{\frac{1}{2}}} e^{t\delta}I^a(t)dt+e^{-T\delta}\int_{\mathscr{T}-B_{\frac{1}{2}}} e^{t\delta}I^b(t)dt \\ &\text{and} \\ x^2(T)&=e^{-T\delta}x_0+e^{-T\delta}\int_{B_{\frac{1}{2}}} e^{t\delta}I^b(t)dt+e^{-T\delta}\int_{\mathscr{T}-B_{\frac{1}{2}}} e^{t\delta}I^a(t)dt \end{aligned}\right\} \tag{2.15}$$

On the other hand, since P^a and P^b are the terminal states corresponding to the investment policies $I^a(t)$ and $I^b(t)$ respectively, we have

$$\left.\begin{aligned} P^a&=e^{-T\delta}x_0+e^{-T\delta}\int_{\mathscr{T}} e^{t\delta}I^a(t)dt \\ &\text{and} \\ P^b&=e^{-T\delta}x_0+e^{-T\delta}\int_{\mathscr{T}} e^{t\delta}I^b(t)dt \end{aligned}\right\} \tag{2.16}$$

Taking account of (2.14) we obtain the following equality from (2.15) and (2.16)

$$x^1(T)=x^2(T)=\frac{1}{2}(P^a+P^b). \tag{2.17}$$

Since the control process (2.3) is normal (*Proposition 3*), the first equality of (2.17) implies that $I^1(t)=I^2(t)$ almost everywhere in $\mathscr{T}$. And this implies that $I^a(t)=I^b(t)$ almost everywhere in the subintervals $B_{\frac{1}{2}}$ and $(\mathscr{T}-B_{\frac{1}{2}})$. However this contradicts the above supposition that P^a and P^b are distinct terminal states on the line segment l. Therefore the set of attainability $A(T)$ is strictly convex in R^n. (Q.E.D.)

In the above discussions we have proved that the set of all terminal states of the control process (2.3) with the initial state of capital stocks x_0 and the constraint on the investment policy (2.5) is a compact and strictly convex set in R^n, and that each terminal state represented as a point in the boundary of the set of attainability is attained by a unique extremal investment policy. And we have shown what the extremality condition is.

Now we introduce the objective functional (2.2) into the discussion. For that purpose define a new state variable $x^0(t)$ by

$$x^0(t)=\int_0^t \{S[I(\tau),\ \tau]-\Pi[x(\tau),\ \tau]\}e^{-r\tau}d\tau \tag{2.18}$$

and

$$x^0(0)=0. \tag{8.19}$$

And define a $(n+1)$-vector $\hat{x}(t)$ as

$$\hat{x}(t)=\begin{bmatrix} x^0(t) \\ x(t) \end{bmatrix}$$

and the initial state vector as

$$\hat{x}_0=\begin{bmatrix}0\\ x_0\end{bmatrix}.$$

Then we have $J(I)=x^0(T)$, and so the aim of the firm is to minimize the value of $x^0(t)$ at $t=T$. The state variable $x^0(t)$ is controlled by the process

$$\dot{x}^0=\{S[I(t),\ t]-\Pi[x(t),\ t]\}e^{-rt} \tag{2.20}$$

Let $\hat{A}(T)$ denote the set of attainability of the extended control process (2.20) and (2.3) with initial state $\hat{x}_0$ and constraint set Ω. The set $\hat{A}(T)$ consist of all endpoints of the trajectory $\hat{x}(t)$ initiating at $\hat{x}(0)=\hat{x}_0$, and the trajectory $\hat{x}(t)$ is the solution of the differential system (2.20) and (2.3) corresponding to the investment policy $I(t)\subset\Omega$ on $0\leqq t\leqq T$.

Now investment policies $I(t)$ on $0\leqq t\leqq T$ have values in a compact convex set $\Omega\subset R^n$, so the set of attainability $\hat{A}(T)$ is bounded in R^{n+1}. The projection of $\hat{A}(T)$ on the x-space R^n is the compact and strictly convex set $A(T)$ defined previously. In the following discussion we examine geometrical properties of $\hat{A}(T)$. When we measure $x^0(T)$ vertically, we need not pay attention to the upper boundary of $\hat{A}(T)$, and only the lower boundary is significant because we seek the investment policy that minimizes $x^0(T)$.

For the extended process (2.20) and (2.3), an investment policy $I(t)\subset\Omega$ on $0\leqq t\leqq T$ which steers $\hat{x}_0$ to a point in the lower boundary of $\hat{A}(T)$ is now called extremal investment policy. Let $\hat{A}_v$ denote the set of all points $(x^0,\ x)\in R^{n+1}$ for which there exists a point $(\chi^0,\ x)\in\hat{A}(T)$ with $\chi^0\leqq x^0$.[9]

Then we can prove the existence of the optimal investment policy on $0\leqq t\leqq T$.

Proposition 5. Consider the extended control process

$$\dot{x}^0=\{S[I(t),\ t]-\Pi[x(t),\ t]\}e^{-rt} \tag{2.20}$$

$$\dot{x}=I(t)-\delta x(t) \tag{2.3}$$

with initial state $\hat{x}_0$ and constraint on the investment policy

$$0\leqq I(t)\leqq I_{\max}\qquad (0\leqq t\leqq T). \tag{2.5}$$

Then the set $\hat{A}_v\subset R^{n+1}$ is closed and convex, and the lower boundary of $\hat{A}_v$ belongs to the set of attainability $\hat{A}(T)$. The lower boundary of $\hat{A}(T)$ constitutes a strictly convex hypersurface defined over the set of attainability of capital stocks $A(T)$. The minimum value of the objective functional is obtained at a unique point on the lower boundary of the set $\hat{A}_v$.

Proof. First we prove that the set $\hat{A}_v$ is closed. Consider a sequence

(9) Therefore the lower boundary of $\hat{A}_v$ is the lower boundary of $\hat{A}(T)$.

of points $\{\hat{a}^{\nu}\}$ which converges to the point $\hat{\bar{a}}$ in R^{n+1}, where

$$\hat{a}^{\nu}=\begin{bmatrix} a^{0\nu} \\ a^{\nu} \end{bmatrix} \text{ and } \hat{\bar{a}}=\begin{bmatrix} \bar{a}^{0} \\ \bar{a} \end{bmatrix}.$$

From the definition of $\hat{A}_{v}$, we can find a sequence of investment policies $\{I^{\nu}(t)\}$ ($I^{\nu}(t)\subset\Omega$ on $0\leqq t\leqq T$) with a sequence of the trajectories $\{\hat{x}^{\nu}(t)\}$ such that $x^{\nu}(T)=a^{\nu}$ and $x^{0\nu}(T)\leqq a^{0\nu}$, where

$$\hat{x}^{\nu}(t)=\begin{bmatrix} x^{0\nu}(t) \\ x^{\nu}(t) \end{bmatrix}.$$

Moreover we can suppose, as shown in the proof of *Proposition 1*, that (a subsequence denoted by the same symbol) $\{I^{\nu}(t)\}$ converges weakly to an investment policy $\bar{I}(t)\subset\Omega$ and the corresponding sequence of the trajectories of capital stocks $\{x^{\nu}(t)\}$ converges to $\bar{x}(t)$, that is,

$$\lim_{\nu\to\infty} I^{\nu}(t)=\bar{I}(t)\subset\Omega$$

and

$$\lim_{\nu\to\infty} x^{\nu}(t)=\bar{x}(t)$$

Therefore we obtain the inequality

$$\lim_{\nu\to\infty} a^{0\nu}=\bar{a}^{0}\geqq\liminf_{\nu\to\infty}\ x^{0\nu}(T). \tag{2.21}$$

By the fact that the set Ω is a compact convex set in the I-space R^{n} and the set $\hat{A}(T)$ is bounded in the (x^{0}, x)-space R^{n+1}, we have

$$\liminf_{\nu\to\infty}\ x^{0\nu}(T)\geqq\bar{x}^{0}(T) \tag{2.22}$$

Therefore from (2.21) and (2.22), we can conclude that the terminal state of the trajectory $\hat{\bar{x}}(t)$ corresponding to the investment policy $\bar{I}(t)$ is $\hat{\bar{x}}(T)$, where

$$\hat{\bar{x}}(t)=\begin{bmatrix} \bar{x}^{0}(t) \\ \bar{x}(t) \end{bmatrix}$$

and

$$\hat{\bar{x}}(T)=\begin{bmatrix} \bar{x}^{0}(T) \\ \bar{x}(T) \end{bmatrix}.$$

Thus the terminal state $\hat{\bar{a}}$ belongs to $\hat{A}_{v}$, and so $\hat{A}_{v}$ is a closed set in R^{n+1}.

If the terminal state $\hat{\bar{a}}$ belongs to the lower boundary of the set $\hat{A}(T)$, then we have $\bar{x}^{0}(T)=\bar{a}^{0}$ and $\bar{x}(T)=\bar{a}$, and so the investment policy $\bar{I}(t)$ steers $\hat{x}^{0}$ to $\hat{\bar{a}}$. Therefore the lower boundary of the set $\hat{A}_{v}$ belongs to the set $\hat{A}(T)$.

Next we show the strict convexity of the lower boundary of $\hat{A}(T)$. Consider two extremal investment policies $I^1(t)$ and $I^2(t)\subset\Omega$ on $0\leqq t\leqq T$ each of which steers $\hat{x}_0$ to some point on the lower boundary of $\hat{A}(T)$. Let the terminal states corresponding to $I^1(t)$ and $I^2(t)$ be denoted by $\hat{x}^1(T)$ and $\hat{x}^2(T)$ respectively. And define a point $\hat{P}$ by

$$\hat{P}=\mu\hat{x}^1(T)+(1-\mu)\hat{x}^2(T) \qquad (0<\mu<1),$$

where

$$\hat{P}=\begin{bmatrix} P^0 \\ P \end{bmatrix} \text{ and } \hat{x}^i(T)=\begin{bmatrix} x^{0i}(T) \\ x^i(T) \end{bmatrix} \qquad (i=1,\ 2).$$

To prove that the lower boundary of the set $\hat{A}(T)$ is a strictly convex hypersurface, we have to construct an admissible investment policy which steers $\hat{x}_0$ to $\hat{P}$ and to show that the point $\hat{P}$ is an interior point of $\hat{A}(T)$.

Now define an investment policy $I^\mu(t)$ on $0\leqq t\leqq T$ by

$$I^\mu(t)=\mu I^1(t)+(1-\mu)I^2(t),$$

and let $x^{0\mu}(t)$ and $x^\mu(t)$ be the corresponding trajectories. Then the terminal state of capital stocks is given by

$$x^\mu(T)=\mu x^1(T)+(1-\mu)x^2(T)=P \tag{2.23}$$

as shown in the proof of *Proposition 1.*

As for $x^{0\mu}(T)$ we obtain

$$\begin{aligned} x^{0\mu}(T)&=\int_0^T\{S[I^\mu(t),\ t]-\Pi[x^\mu(t),\ t]\}e^{-rt}\ dt \\ &=\int_0^T\{S[\mu I^1(t)+(1-\mu)I^2(t),\ t]-\Pi[\mu x^1(t)+(1-\mu)x^2(t),\ t]\}e^{-rt}\,dt. \end{aligned}$$

Since the expansion cost function S is strictly convex in I and the operating profit function Π is concave in x, we obtain the inequality

$$\begin{aligned} &S[\mu I^1(t)+(1-\mu)I^2(t),\ t]-\Pi[\mu x^1(t)+(1-\mu)x^2(t),\ t] \\ &\quad<\mu\{S[I^1(t),\ t]-\Pi[x^1(t),\ t]\}+(1-\mu)\{S[I^2(t),\ t]-\Pi[x^2(t),\ t]\}. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\int_0^T\{S[\mu I^1(t)+(1-\mu)I^2(t),\ t]-\Pi[\mu x^1(t)+(1-\mu)x^2(t),\ t]\}e^{-rt}\ dt \\ &\quad<\mu\int_0^T\{S[I^1(t),\ t]-\Pi[x^1(t),\ t]\}e^{-rt}\ dt+(1-\mu)\int_0^T\{S[I^2(t),\ t] \\ &\qquad-\Pi[x^2(t),\ t]\}e^{-rt}\ dt \\ &\quad=\mu x^{01}(T)+(1-\mu)x^{02}(T) \\ &\quad=P^0. \end{aligned}$$

So we have

$$x^{0\mu}(T)<P^0. \tag{2.24}$$

Since values of investment policies $I^1(t)$ and $I^2(t)$ are in the compact convex set Ω on the planning period $0 \leqq t \leqq T$, the corresponding terminal states $x^1(T)$ and $x^2(T)$ belong to the set of attainability of capital stocks $A(T)$. Therefore from (2.23) and (2.24) and the fact that terminal state P^0 is given as a positive convex linear combination of $x^{01}(T)$ and $x^{02}(T)$, we find that the terminal state $\hat{P}$ is an interior point of $\hat{A}(T)$.

By the above discussions the lower boundary of the set $\hat{A}(T)$ is a strictly convex hypersurface defined over $A(T)$.

Since the lower boundary of $\hat{A}(T)$ is a strictly convex hypersurface, the objective functional $J(I)=x^0(T)$ achieves its minimum value at a unique point on the boundary. And this point is attainable, so our problem has the optimal investment policy. (Q.E.D.)

An optimal investment policy steers the initial state $\hat{x}^0$ to a terminal state in the lower boundary of the set of attainability $\hat{A}(T)$. Therefore an optimal investment policy must be extremal for the extended process (2.20) and (2.3).

Then we proceed to show what the extremality condition is.

Proposition 6. Consider the extended control process

$$\dot{x}^0=\{S[I(t),\ t]-\Pi[x(t),\ t]\}e^{-rt} \tag{2.20}$$

$$\dot{x}=I(t)-\delta x(t) \tag{2.3}$$

with initial state $\hat{x}^0$ and constraint on the investment policy

$$0 \leqq I(t) \leqq I^{\max} \qquad (0 \leqq t \leqq T). \tag{2.5}$$

An investment policy $\bar{I}(t) \subset \Omega$ on $0 \leqq t \leqq T$ with the corresponding trajectory $\hat{\bar{x}}(t)$ is extremal if and only if there exists a nonvanishing $(n+1)$—vector $\hat{\eta}(t)=[\eta^0\ \eta(t)]$ satisfying the differential system

$$\left.\begin{aligned} &\dot{\eta}^0=0,\ \eta^0 \leqq 0 \\ &\dot{\eta}=\eta^0\frac{\partial \Pi[\bar{x}(t),\ t]}{\partial x}e^{-rt}+\eta(t)\delta \end{aligned}\right\} \tag{2.25}$$

and satisfying the maximum condition

$$\eta^0 S[\bar{I}(t),\ t]e^{-rt}+\eta(t)\bar{I}(t)=\max_{I\in\Omega}\ [\eta^0 S(I,\ t)e^{-rt}+\eta(t)I] \tag{2.26}$$

almost everywhere on the planning period $0 \leqq t \leqq T$, where

$$\hat{\bar{x}}(t)=\begin{bmatrix}\bar{x}^0(t)\\ \bar{x}(t)\end{bmatrix}.$$

Proof. First, assuming that the investment policy $\bar{I}(t) \subset \Omega$ on $0 \leqq t \leqq T$, the corresponding trajectory $\hat{\bar{x}}(t)$, and the adjoint trajectory $\hat{\eta}(t)$ satisfy the differential system (2.20), (2.3) and (2.25), and the maximum condition (2.26) almost everywhere, we prove that the investment policy $\bar{I}(t)$ steers $\hat{x}_0$ to the lower boundary of the set of attainability $\hat{A}(T)$. For that purpose it suffices to obtain the inequality

$$\hat{\eta}(T)\hat{\bar{x}}(T) \geqq \hat{\eta}(T)\hat{x}(T), \tag{2.27}$$

where $\hat{x}(T)$ is the terminal state of the trajectory $\hat{x}(t)$ corresponding to any admissible investment policy $I(t) \subset \Omega$ on $0 \leqq t \leqq T$, and

$$\hat{x}(t) = \begin{bmatrix} x^0(t) \\ x(t) \end{bmatrix}.$$

The inequality (2.27) implies (1) if $\eta^0 < 0$, then $\hat{\bar{x}}(T)$ belongs to the lower boundary of the set $\hat{A}(T)$, and (2) if $\eta^0 = 0$, then $\hat{\bar{x}}(T)$ belongs to the lateral boundary of the set $\hat{A}_v$. However if $\eta^0 = 0$, then the trajectory of capital stocks $\bar{x}(t)$ corresponding to $\bar{I}(t)$ is extremal in the sense of *Proposition 2*, and so the terminal state $\bar{x}(T)$ lies on the boundary of the set $A(T)$. Moreover the control process (2.3) is normal (*Proposition 3*), $\bar{I}(t)$ is the only investment policy that steers x_0 to the terminal state $\bar{x}(T)$ in $\partial A(T)$. Therefore the terminal state $\hat{\bar{x}}(T)$ is the unique point of the set $\hat{A}(T)$ that lies in the vertical direction of $\bar{x}(T)$. Thus $\hat{\bar{x}}(T)$ belongs to the lower boundary of $\hat{A}(T)$ in all cases, and so the investment policy $\bar{I}(T)$ is extremal.

We must now prove the inequality (2.27). Differentiate the inner product $\hat{\eta}(t)\hat{x}(t)$ with respect to t, and we obtain

$$\frac{d}{dt}[\hat{\eta}(t)(\hat{x}t)] = \dot{\eta}^0 x^0 + \eta^0 \dot{x}^0 + \eta\dot{x} + \dot{\eta}x.$$

Using differential systems (2.20), (2.3) and (2.5) we have

$$\begin{aligned}
&\hat{\eta}(T)\hat{x}(T) - \hat{\eta}(0)\hat{x}_0 \\
&= \int_0^T [\dot{\eta}^0 x^0 + \eta^0 \dot{x}^0 + \dot{\eta}x + \eta\dot{x}]dt \\
&= \int_0^T \{\eta^0[S(I(t),\ t) - \Pi(x(t),\ t)]e^{-rt} + \eta(t)[I(t) - \delta x(t)] \\
&\quad + \left[\eta^0 \frac{\partial \Pi(\bar{x},\ t)}{\partial x} e^{-rt} + \eta(t)\delta\right] x(t)\}dt \\
&= \int_0^T \{\eta^0 \left[\frac{\partial \Pi(\bar{x},\ t)}{\partial x} x(t) - \Pi(x(t),\ t)\right] e^{-rt} \\
&\quad + \eta^0 S[I(t),\ t]e^{-rt} + \eta(t)I(t)\}dt.
\end{aligned} \tag{2.28}$$

Now specialize $I(t)$ to $\bar{I}(t)$ and $x(t)$ to $\bar{x}(t)$ to obtain

$$\hat{\eta}(T)\hat{\bar{x}}(T) - \hat{\eta}(0)\hat{x}_0$$

$$= \int_0^T \{\eta^0 \left[\frac{\partial \Pi(\bar{x}, t)}{\partial x} \bar{x}(t) - \Pi(\bar{x}(t), t) \right] e^{-rt}$$
$$+ \eta^0 S[\bar{I}(t), t] e^{-rt} + \eta(t) \bar{I}(t) \} dt. \tag{2.29}$$

The investment policy $\bar{I}(t)$ satisfies the maximum condition (2.26) almost everywhere, and so we have

$$\eta^0 S[I(t), t] e^{-rt} + \eta(t) I(t) \leqq \eta^0 S[\bar{I}(t), t] e^{-rt} + \eta(t) \bar{I}(t) \tag{2.30}$$

almost everywhere on the planning period. We have assumed the concavity of the operating profit function Π in x, and so we obtain the inequality

$$\frac{\partial \Pi[\bar{x}, t]}{\partial x} \bar{x}(t) - \Pi[\bar{x}(t), t] \leqq \frac{\partial \Pi[\bar{x}, t]}{\partial x} x(t) - \Pi[x(t), t]. \tag{2.31}$$

Therefore from (2.28), (2.29), (2.30) and (2.31) the inequality (2.27) is obtained.

Conversely assume that the investment policy $\bar{I}(t) \subset \Omega$ with the corresponding trajectory $\hat{\bar{x}}(t)$ is extremal, and so steers $\hat{x}_0$ to a terminal state $\hat{\bar{x}}(T)$ on the lower boundary of the set of attainability $\hat{A}(T)$.

Let $\hat{\eta}(T) = [\eta^0 \ \eta(T)]$ be an outward normal vector to the set $\hat{A}(T)$ at the point $\hat{\bar{x}}(T)$. Then $\eta^0 \leqq 0$, and $\eta^0 = 0$ just in case $\bar{x}(T)$ lies on the boundary of the set of attainability $A(T)$. Define $\hat{\eta}(t) = [\eta^0 \ \eta(t)]$ as the solution of the differential system (2.25) under the terminal condition $\hat{\eta}(T)$. Then we must prove that the maximum condition (2.26) is obtained for almost all t in the planning period.

In the case of $\eta^0 = o$, $\eta(T)$ is an outward normal vector to the set $A(T)$ at the point $\bar{x}(T)$ in $\partial A(T)$, and the maximum condition

$$\eta(t) \bar{I}(t) = \max_{I \in \Omega} \eta(t) I$$

holds almost everywhere on the planning period $0 \leqq t \leqq T$ as proved in *Proposition 2.*

Next we consider the case of $\eta^0 < 0$. In this case we can assume $\eta^0 = -1$ without loss of generality. Assume that the investment policy $\bar{I}(t) \subset \Omega$ on $0 \leqq t \leqq T$ does not satisfy the maximum condition (2.26) on some subinterval of positive duration in the planning period, and define an investment policy $\tilde{I}(t) \subset \Omega$ on $0 \leqq t \leqq T$ by

$$-S[\tilde{I}(t), t] e^{-rt} + \eta(t) \tilde{I}(t) = \max_{I \in \Omega} [-S(I, t) e^{-rt} + \eta(t) I].$$

Let K denote a compact subinterval of positive duration in $0 < t < T$ whereon investment policies $\bar{I}(t)$ and $\tilde{I}(t)$ are continuous and where

$$-S[\bar{I}(t), t] e^{-rt} + \eta(t) \bar{I}(t) < -S[\tilde{I}(t), t] e^{-rt} + \eta(t) \tilde{I}(t) - \gamma$$

for some constant $\gamma>0$. Take a time t_1 in K for which the interval $(t_1, t_1+\varepsilon)\cap K$ has a measure $\varepsilon[1+0(\varepsilon)]$ for all small $\varepsilon>0$, where $\lim_{\varepsilon\to 0} 0(\varepsilon)=0$. Define an investment policy $I^\varepsilon(t)$ as

$$I^\varepsilon(t)=\begin{cases}\tilde{I}(t) & \text{on } K\cap(t_1, t_1+\varepsilon)\\ \bar{I}(t) & \text{elsewhere on } [0, T]\end{cases}$$

Then for sufficiently small $\varepsilon>0$, the trajectory $\hat{x}^\varepsilon(t)$ corresponding to the investment policy $I^\varepsilon(t)$ approximates uniformly the trajectory $\hat{\bar{x}}(t)$ corresponding to $\bar{I}(t)$, that is,

$$|\hat{x}^\varepsilon(t)-\hat{\bar{x}}(t)|<k\varepsilon$$

for some $k>0$ and on the planning period $0\leqq t\leqq T$, where

$$\hat{x}^\varepsilon(t)=\begin{bmatrix}x^{0\varepsilon}(t)\\ x^\varepsilon(t)\end{bmatrix}.$$

Since $(\partial\Pi/\partial x)e^{-rt}$ is continuous, we obtain

$$\left|\frac{\partial\Pi(\bar{x}, t)}{\partial x}[x^\varepsilon(t)-\bar{x}(t)]e^{-rt}-\Pi[x^\varepsilon(t), t]e^{-rt}+\Pi[x(t), t]e^{-rt}\right|<\varepsilon 0(\varepsilon).$$

Using the computations in the first half of this proof, we have

$$\begin{aligned}&\hat{\eta}(T)\hat{x}^\varepsilon(T)-\hat{\eta}(0)\hat{x}_0\\ &=\int_0^T\left\{-\left[\frac{\partial\Pi(\bar{x}, t)}{\partial x}x^\varepsilon(t)-\Pi(x^\varepsilon(t),t)\right]e^{-rt}-S[I^\varepsilon(t), t]e^{-rt}+\eta(t)I^\varepsilon(t)\right\}dt.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\hat{\eta}(T)\hat{\bar{x}}(T)-\hat{\eta}(T)\hat{x}^\varepsilon(T)\\ &\leqq\int_0^T\left\{\frac{\partial\Pi(\bar{x}, t)}{\partial x}[x^\varepsilon(t)-\bar{x}(t)]-\Pi[x^\varepsilon(t), t]+\Pi[\bar{x}(t), t]\right\}e^{-rt}dt\\ &\quad-\gamma\varepsilon[1+o(\varepsilon)].\end{aligned}$$

Then, for sufficiently small $\varepsilon>0$, the inequality

$$\hat{\eta}(T)\hat{\bar{x}}(T)<\hat{\eta}(T)\hat{x}^\varepsilon(T)$$

is obtained. But this is impossible since $\hat{\eta}(T)$ is an outward normal vector to $\hat{A}(T)$ at $\hat{\bar{x}}(T)$ in the lower boundary of $\hat{A}(T)$. Therefore the extremal investment policy $\bar{I}(t)$ must satisfy the maximum condition (2.26) with the adjoint trajectory $\hat{\eta}(t)$ almost everywhere on the planning period $0\leqq t\leqq T$. (Q.E.D.)

An optimal investment policy must be extremal and so it must satisfy the conditions stated in *Proposition 6*. We prove then the uniqueness of the optimal investment policy.

Proposition 7. Consider the extended control process

$$\dot{x}^0=\{S[I(t),\ t]-\Pi[x(t),\ t]\}e^{-rt} \tag{2.20}$$

$$\dot{x}=I(t)-\delta x(t) \tag{2.3}$$

with initial state $\hat{x}_0$ and constraint on the investment policy

$$0\leqq I(t)\leqq I_{\max} \qquad (0\leqq t\leqq T). \tag{2.5}$$

Any two extremal investment policies which steer the initial state $\hat{x}_0$ to the same terminal state on the lower boundary of the set of attainability $\hat{A}(T)$ must coincide almost everywhere on the planning period $0\leqq t\leqq T$. Moreover, there exists a unique optimal investment policy.

Proof. Consider two extremal investment policies $I^1(t)$ and $I^2(t)\subset\Omega$ on $0\leqq t\leqq T$ both of which steer $\hat{x}_0$ to the same point $\hat{\bar{x}}(T)$ on the lower boundary of $\hat{A}(T)$, where

$$\hat{\bar{x}}(T)=\begin{bmatrix}\bar{x}^0(t)\\ \bar{x}(t)\end{bmatrix}.$$

Let $\hat{\eta}(t)=[\eta^0\ \eta(t)]$ be the corresponding adjoint trajectory.

If an $(n+1)$-vector $[0\ \eta(T)]$ determines an outward normal vector to $\hat{A}(T)$ at $\hat{\bar{x}}(T)$, then, as proved in *proposition 3*, we have $I^1(t)=I^2(t)$ almost everywhere on $0\leqq t\leqq T$.

Next we consider the case of $\eta^0<0$. In this case we may assume $\eta^0=-1$ without loss of generality. Let $\hat{\bar{x}}(t)$ denote the corresponding trajectory to the investment policy $I^1(t)\subset\Omega$ on $0\leqq t\leqq T$, and let $\hat{\eta}^1(t)=[-1\ \eta^1(t)]$ be the corresnd-ing adjoint trajectory, such that $\hat{\eta}^1(T)$ is an outward normal vector to $\hat{A}(T)$ at $\hat{\bar{x}}(T)$. Since the investment policy $I^1(T)$ is extremal, we have

$$-S[I^1(t),\ t]e^{-rt}+\eta^1(t)I^1(t)=\max_{I\in\Omega}\ [-S(I,\ t)e^{-rt}+\eta^1(t)I]\equiv\omega(t)$$

almost everywhere on the planning period $0\leqq t\leqq T$.

Let $\hat{x}^2(t)$ denote the corresponding trajectory to the investment policy $I^2(t)$, and suppose that $I^2(t)$ steers $\hat{x}_0$ to the same boundary point $\hat{\bar{x}}(T)$.

If, for the specified $\eta^1(t)$, the investment policy $I^2(t)$ fails to satisfy the maximum condition on some subinterval of $[0,\ T]$ with a positive duration, we obtain the inequality

$$\int_0^T\{-S[I^1(t),\ t]e^{-rt}+\eta^1(t)I^1(t)\}dt>\int_0^T\{-S[I^2(t),\ t]e^{-rt}+\eta^1(t)I^2(t)\}dt. \tag{2.32}$$

Now compute $\hat{\eta}^1(T)\hat{\bar{x}}(T)-\hat{\eta}^1(T)\hat{x}^2(T)$ by the same method in the proof of *proposition 6.* From the inequality (2.32) and the assumption that the operating profit function Π is concave in x. we obtain

$$\hat{\eta}^1(T)\hat{\hat{x}}(T)>\hat{\eta}^1(T)\hat{x}^2(T)=\hat{\eta}^1(T)\hat{\hat{x}}(T).$$

However this is a contradition. Therefore we can say that any two extremal investment policies $I^1(t)$ and $I^2(t)$ must satisfy the same maximum condition almost everywhere on $0\leqq t\leqq T$.

Next, consider the investment policy $\frac{1}{2}[I^1(t)+I^2(t)]$. Then by the strict convexity of the expansion cost functions S in I we obtain, whenever $I^1(t)\neq I^2(t)$,

$$-S[\tfrac{1}{2}(I^1(t)+I^2(t)),\ t]e^{-rt}+\eta^1(t)\tfrac{1}{2}[I^1(t)+I^2(t)]>\tfrac{1}{2}\omega(t)+\tfrac{1}{2}\omega(t)$$

Thus we conclude that two investment policies $I^1(t)$ and $I^2(t)$ must coincide almost everywhere on the planning period.

Since the objective functional $J(I)=x^0(T)$ achieves its minimum value at a unique point of the lower boundary of $\hat{A}(T)$, as proved in *proposition 5*, there exists a unique optimal investment policy. (Q.E.D.)

We proceed now to show what the optimality condition is.

Proposition 8. Consider the extended control process

$$\dot{x}^0=\{S[I(t),\ t]-\Pi[x(t),\ t]\}e^{-rt} \tag{2.20}$$

$$\dot{x}=I(t)-\delta x(t) \tag{2.3}$$

with the initial state $\hat{x}_0$ and the constraint on the investment policy

$$0\leqq I(t)\leqq I_{\max} \qquad (0\leqq t\leqq T). \tag{2.5}$$

An investment policy $I^*(t)\subset\Omega$ on $0\leqq t\leqq T$ with corresponding trajectory $\hat{x}^*(t)$ is optimal if $I^*(t)$, $x^*(t)$ and nonvanishing $\eta(t)$ satisfy the differential system

$$\dot{\eta}=-\frac{\partial\Pi(x^*,\ t)}{\partial x}e^{-rt}+\eta(t)\delta,\quad \eta(T)=0 \tag{2.33}$$

and satisfy the maximum condition

$$-S[I^*(t),\ t]e^{-rt}+\eta(t)I^*(t)=\max_{I\in\Omega}\ [-S(I,\ t)e^{-rt}+\eta(t)I] \tag{2.34}$$

almost everywhere on the planning period $0\leqq t\leqq T$.

Proof. Assume that the investment policy $I^*(t)\subset\Omega$ on $0\leqq t\leqq T$, the corresponding trajectory of capital stocks $x^*(t)$, and the corresponding adjoint trajectory $\eta(t)$ satisfy the maximum condition (2.34) and the differential system (2.33) with the transversality condition $\eta(T)=0$. Let $\tilde{I}(t)$ denote any other admissible investment policy, and let $\tilde{x}(t)$ be the corresponding trajectory of capital stocks.

Differentiate $-x^0(t)+\eta(t)x(t)$ with respect to t, and compute $-x^{0*}(T)+$

$\tilde{x}^0(T)$ under the transversality condition $\gamma(T)=0$. Then we obtain

$$-x^{0*}(T)+\tilde{x}^0(T)$$
$$=\int_0^T\left\{\frac{\partial\Pi(x^*,\ t)}{\partial x}[\tilde{x}(t)-x^*(t)]e^{-rt}+[\Pi(x^*(t),\ t)-\Pi(\tilde{x}(t),\ t)]e^{-rt}\right.$$
$$+[-S(I^*(t),\ t)e^{-rt}+\gamma(t)I^*(t)]-[-S(\tilde{I}(t),\ t)e^{-rt}$$
$$\left.+\gamma(t)\tilde{I}(t)]\right\}dt \tag{2.35}$$

The concavity of the operating profit function Π in x shows that

$$\frac{\partial\Pi[x^*,\ t]}{\partial x}[\tilde{x}(t)-x^*(t)]\geqq\Pi[\tilde{x}(t),\ t]-\Pi[x^*(t),\ t] \tag{2.36}$$

The investment policy $I^*(t)$ satisfies the maximum condition (2.34) almost everywhere on $0\leqq t\leqq T$. On the other hand the investment policy $\tilde{I}(t)$ is distinct from $I^*(t)$, and so there exists a subinterva of a positive duration of $[0,\ T]$ whereon the inequality

$$-S[\tilde{I}(t),\ t]e^{-rt}+\gamma(t)\tilde{I}(t)<\max_{I\in\Omega}\ [-S(I,\ t)e^{-rt}+\gamma(t)I]$$

holds for $\gamma(t)$ that satisfies the maximum cndition (2.34) together with $I^*(t)$.(10) Therefore we obtain the inequality

$$\int_0^T\{-S[I^*(t),\ t]e^{-rt}+\gamma(t)I^*(t)\}dt>\int_0^T\{-S[\tilde{I}(t),\ t]e^{-rt}$$
$$+\gamma(t)\tilde{I}(t)\}dt. \tag{2.37}$$

Thus, from (2.36) and (2.37), the right hand side of (2.35) is positive. So we have

$$x^{0*}(T)<\tilde{x}^0(T)$$

as required. (Q.E.D.)

By the above arguments the uniqueness of the optimal investment policy and the optimality conditions were shown. Next, we prove that the corresponding optimal trajectory of capital stocks and the corresponding adjoint trajectory are uniquely determined respectively.

Proposition 9. Consider the extended control process

$$\dot{x}^0=\{S[I(t),\ t]-\Pi[x(t),\ t]\}e^{-rt} \tag{2.20}$$

$$\dot{x}=I(t)-\delta x(t) \tag{2.3}$$

(10) Otherwise we have $I^*(t)=\tilde{I}(t)$ almost everywhere on $0\leqq t\leqq T$.

with the initial state $\hat{x}_0$ and the constraint on the investment policy

$$0 \leqq I(t) \leqq I_{\max} \qquad (0 \leqq t \leqq T). \tag{2.5}$$

Then the differential system

$$\dot{x} = I^*(t,\eta) - \delta x(t)$$

$$\dot{\eta} = -\frac{\partial \Pi(x, t)}{\partial x} e^{-rt} + \eta(t)\delta, \qquad \eta(T) = 0$$

has a unique solution $x^*(t)$, $\eta^*(t)$. Here $I^*(t, \eta)$ is determined by the maximum condition

$$-S[I^*(t), t]e^{-rt} + \eta(t)I^*(t) = \max_{I \in \Omega} [-S(I, t)e^{-rt} + \eta(t)I], \tag{2.34}$$

and $I^*(t) = I^*(t, \eta)$ with the corresponding trajectory $x^*(t)$ is the unique optimal investment policy.

Proof. Let $I^*(t)$ be the optimal investment policy that steers the initial state $\hat{x}_0$ to a terminal state $\hat{x}^*(T)$ by the corresponding trajectory $\hat{x}^*(t)$, where $\hat{x}^*(t)$ is defined as

$$\hat{x}^*(t) = \begin{bmatrix} x^{0*}(t) \\ x^*(t) \end{bmatrix}.$$

Then the terminal state $\hat{x}^*(T)$ belongs to the lowIr boundary of the set of attainability $\hat{A}(T)$, and the inequality $x^{0*}(T) < x^0(T)$ holds for any other attainable $x^0(T)$. And there exists a supporting hyperplane p^* of $\hat{A}(T)$ at $\hat{x}^*(T)$ which is parallel to $x^0 = 0$.

Therefore we can define the vector $\hat{\eta}^*(T) = [\eta^{0*}(T)\ \eta^*(T)[=[-1\ 0]$ as an outward normal vector to $\hat{A}(T)$ at $\hat{x}^*(T)$.

Now let $\eta^*(t)$ be the solution of the differential system

$$\dot{\eta} = -\frac{\partial \Pi(x^*, t)}{\partial x} e^{-rt} + \eta(t)\delta$$

under the terminal condition $\eta(T) = 0$, then the investment policy $I^*(t)$ with the corresponding adjoint trajectory $\eta^*(t)$ satisfies the maximum condition (2.34) almost everywhere on $0 \leqq t \leqq T$ as proved in *Proposition 6*. Therefore we have $I^*(t) = I^*(t, \eta)$.

The expansion cost function S is strictly convex in I(11), and the supporting hyperplane p^* meets $\hat{A}(T)$ at only one point $\hat{x}^*(T)$.

Therefore $I^*(t)$, $x^*(t)$, and $\eta^*(t)$ are uniquely determined respectively.

(Q.E.D.)

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(11) $\hat{A}(T)$ is a strictly convex hypersurface defined over the set of attainability of capital stocks $A(T)$.

Now let us seek the optimal investment policy that minimizes the objective functional $J(I)=x^0(T)$. By the above arguments the optimal investment policy, the corresponding optimal trajectory and the corresponding adjoint trajectory are uniquely determined as $I^*(t) \subset \Omega$ on $0 \leqq t \leqq T$, $x^*(t)$ and $\eta(t)$ that satisfy the differential system with the boundary conditions $x(0)=x_0$ and $\eta(T)=0$

$$\dot{x}=I(t)-\delta x(t) \tag{2.3}$$

and

$$\dot{\eta}=-\frac{\partial \Pi(x^*,\ t)}{\partial x}e^{-rt}+\eta(t)\delta, \tag{2.33}$$

and satisfy the maximum condition

$$-S[I^*(t),\ t]e^{-rt}+\eta(t)I^*(t)=\max_{I\in\Omega}[-S(I,\ t)e^{-rt}+\eta(t)I] \tag{2.34}$$

almost everywhere on the planning period $0 \leqq t \leqq T$.

Define the Hamiltonian function H as

$$H=\{S[I(t),t]-\Pi[x(t),t]\}e^{-rt}+\eta(t)[I(t)-\delta x(t). \tag{2.38}$$

Then the maximum condition (2.34) is eqiavalent to the maximimization of the function H with respect to I.

Now we introduce a vector-valued function $\varphi(t)$ defined by

$$\varphi(t)=\eta(t)e.^{rt}$$

Then the vector $\varphi(t)$ is interpreted as the imputed price vector of gross investments. The differential system (2.33) is rewritten as

$$\dot{\varphi}=-\frac{\partial \Pi(x^*,\ t)}{\partial x}+\varphi(t)(rE+\delta),$$

where the boundary condition at $t=T$ is $\varphi(T)=0$ and E is an $n\times n$ identity matrix. And the maximum condition (2.34) and the Hamiltonian function (2.38) are respectively written as

$$-S[I^*(t),\ t]+\varphi(t)I^*(t)=\max_{I\in\Omega}\ [-S(I,\ t)+\varphi(t)I]$$

and

$$H=-\{S[I(t),\ t]-\Pi[x(t),\ t]\}e^{-rt}+\varphi(t)[I(t)-\delta x(t)]e^{-rt}$$

Define a function H^{Δ} by

$$H^{\Delta}=-S[I(t),\ t]+\varphi(t)I(t).$$

By maximizing the function H^{Δ} with respect to $I \in \Omega$, the optimal investment policy is obtained as a function of $\varphi(t)$.

Now differentiating the functio H^{Δ} with espect to I we obtain

$$\frac{\partial H^{\Delta}}{\partial I}=-\frac{\partial S}{\partial I}+\varphi(t).$$

When we ignore the constraint on the investment policy (2.5), the first necessary condition for a maximum is

$$-\frac{\partial S}{\partial I}+\varphi(t)=0. \tag{2.39}$$

And the secondary condition is guaranteed by the strict convexity of the expansion cost function S in I.

Taking account of the constraint on the investment policy (2.5), the investment policy $I^*(t)=I^*(t, \varphi)$ that maximizes the function H^{Δ} can be determined.

2.2.2. In this section we consider the case in which the target set at $t=T$ is prescribed as a compact convex set in the x-space R^n. This is the case in which the firm requires the terminal state of capital stocks to lie in some prescribed compact convex set. Except for the target set at the end of the planning period, other factors constituting the moel are assumed to be same as those of Section 2.2.1.(12)

Let the constraint on the terminal state of capital stocks be represented by

$$x_{\min}\leqq x(T)\leqq x_{\max}, \tag{2.40}$$

where n-vectors $x_{\min}$ and $x_{\max}$ are defined as

$$x_{\min}=\begin{bmatrix} x_{1\min} \\ x_{2\min} \\ \vdots \\ x_{n\min} \end{bmatrix} \text{ and } x_{\max}=\begin{bmatrix} x_{1\max} \\ x_{2\max} \\ \vdots \\ x_{n\max} \end{bmatrix},$$

and $x_{i\min}$ and $x_{i\max}$ are positive finite numbers. Let the set of all endpoints that satisfy the constraint (2.40) be denoted by X in the x-space R^n. The set X is the target set at $t=T$.

Now our problem is to choose the investment polict $I(t)\subset\Omega$ on $0\leqq t\leqq T$ that steers the initial state of capitalstocks x_0 to a terminal state in the target set X and minimizes the objective functional $J(I)$.

Introducting the target set at $t=T$ into the picture, the terminal state $x(T)$ must be in the intersection of the set of attainability $A(T)$ and the target set X. Therefore we must choose the investment policy $I(t)\subset\Omega$ on $0\leqq t\leqq T$ that steers x_0 to some point in the set $A(T)\cap X$ and minimizes $J(I)$.

By the above arguments it is obvious that we nst require

$$A(T)\cap X\neq\phi, \tag{2.41}$$

(12) Therefore *Proposition 1, 2, 3* and *4* in the previous section hold in this section too.

exogenous. The effects of the technological progresses occured elsewhere are singificant, but the research and development of production technology by the firm is very important too. Then we examine the optimal research and development policy of the firm in the following.

Now, define the expanision cost founction S as a strictly convex function of I in the same way as before. The operating profit function Π may be regarded as a function of capital stocks, the supply conditions of perfectly variable factors and the production technology. At each moment of time the production technology may be regarded as a function of the accumulated expenditures for research and development of the production technology. Let $y(t)$ be the accumulated expenditures up to time t, and call this "research and development fund", or simply "R & D fund". Then the operating profit function may be denoted by $\Pi[x(t), y(t), t]$, where $y(t)$ is measured in dollar term. Let $I_y(t)$ be the rate of "gross investment in research and development".

Then we have

$$J(I)=\int_0^T \{S[I(t), t]+I_y(t)-\Pi[x(t), y(t), t]\}e^{-rt}\, dt \tag{3.1}$$

as the objective functional to be minimized.

If we assume that $y(t)$ "wears out" at a constant exponential rate δ_y, the evolution of $y(t)$ is determined by

$$\dot{y}=I_y(t)-\delta_y y(t) \tag{3.2}$$

and

$$y(0)=y_0,$$

where y_0 is the initial state of R & D fund.

On the other hand the evolution of capital stocks is determined by

$$\dot{x}=I(t)-\delta x(t) \tag{3.3}$$

and

$$x(0)=x_0,$$

where x_0 is the initial state of capital stocks.

Now we states several assumptions to be maintained in section 3.2.

(1) $I_y(t)$ is a Lesbegue measurable function on $0\leqq t\leqq T$, and its values must satisfy the constraint

$$0\leqq I_y(t)\leqq I_{y\,\max} \qquad (0\leqq t\leqq T)$$

The set of all I_y that satisfy the constraint is denoted by $\Omega_y\subset R^1$.

(2) The operating profit function Π is concave and differentiable with respect to x and y. For every $x>0$ and $y>0$, $\Pi\geqq 0$, $\partial\Pi/\partial x>0$ and $\partial\Pi/\partial y>0$.

In addition, the assumptions (1) and (3) stated in section 2.1 are maintained.

3.2. Optimal Research and Development Policy

Thus our problem is formulated as follows.

Find the combination of

(1) the optimal investment policy $I(t) \subset \Omega$ on $0 \leqq t \leqq T$ that steers x_0 to a terminal state in some prescribed target set by the control process (3.3), and

(2) the optimal research and development policy $I_y(t) \subset \Omega_y$ on $0 \leqq t \leqq T$ that steers y_0 to a terminal state in some prescribed target set by the control process (3.2),

where the combination of optimal policies is the combination of $I(t)$ and $I_y(t)$ for which the objective functional (3.1) achieves the least possible value.

Obviously this problem is similar to the problems in section 2. So we can say as follows.

1. Let $A_y(T)$ be the set of attainability of the control process (3.3) and (3.2) with the initial state x_{y_0}, and the constraint sets Ω and Ω_y, where

$$x_{y_0} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Then the set $A_y(T)$ is a strictly convex and compact set in the (x, y)-space R^{n+1}.

2. The combination of the investment policy and the research and development policy that steers x_{y_0} to each boundary point of the set $A_y(T)$ is unique.

3. Define $x^0(t)$ by

$$\dot{x}^0 = \{S[I(t),\ t] + I_y(t) - \Pi[x(t),\ y(t),\ t]\} e^{-rt} \qquad (3.4)$$

$$x^0(0) = 0,$$

and $(n+2)$-vectors $\hat{x}_y(t)$ and $\hat{x}_{y_0}$ by

$$\hat{x}_y(t) = \begin{pmatrix} x^0(t) \\ x(t) \\ y(t) \end{pmatrix} \text{ and } \hat{x}_{y_0} = \begin{pmatrix} 0 \\ x_0 \\ y_0 \end{pmatrix}.$$

Let $\hat{A}_y(T) \subset R^{n+2}$ denote the set of attainability of the trajectory $\hat{x}_y(t)$ initiating as $\hat{x}_{y_0}$ that corresponds to $I(t) \subset \Omega$ and $I_y(t) \subset \Omega_y$ on $0 \leqq t \leqq T$. Then the lower boundary of the set $\hat{A}_y(T)$ is a strictly convex hypersurface, and the combination of policies that steers $\hat{x}_{y_0}$ to each point on the lower boundary of $\hat{A}_y(T)$ exists. Moreover, there exists a unique point at which the objective functional achieves the least possible value.

4. If the target set at the end of the planning period is the (x, y)-space R^{n+1},

then there exists a combination of the optimal investment policy and the optimal research and development policy. The optimal investment policy $I^*(t) \subset \Omega$ on $0 \leqq t \leqq T$, the optimal research and development policy $I_y^*(t) \subset \Omega$ on $0 \leqq t \leqq T$, the corresponding optimal trajectory $\hat{x}_y^*(t)$, and the corresponding adjoint trajectory $\eta_{xy}(t) = [\eta(t)\ \eta_y(t)]$ are determined by

the differential system (3.4), (3.3) and (3.2) with initial condition $\hat{x}_y(0) = \hat{x}_{y_0}$,

the differential system

$$\left.\begin{aligned} \dot{\eta} &= -\frac{\partial \Pi(x^*, y^*, t)}{\partial x} e^{-rt} + \eta(t)\delta \\ \dot{\eta}_y &= -\frac{\partial \Pi(x^*, y^*, t)}{\partial y} e^{-rt} + \eta_y(t)\delta_y \end{aligned}\right\} \tag{3.5}$$

with terminal condition $\eta_{xy}(T) = 0$,

and the maximum condition

$$-\{S[I^*(t), t] + I_y^*(t)\}e^{-rt} + \eta(t)I^*(t) + \eta_y(t)I_y^*(t)$$
$$= \max_{\substack{I \in \Omega \\ I_y \in \Omega_y}} \{-[S(I, t) + I_y]e^{-rt} + \eta(t)I + \eta_y(t)I_y\}$$

almost everywhere on $0 \leqq t \leqq T$.

5. Let the target set at $t = T$ be prescribed as a compact convex set in R^{n+1}, and assume that the control process (3.3) and (3.2) is controllable, that is, there exists at least one combination of policies with graphs in Ω and Ω_y that steers x_{y_0} to some point in the intersection of $A_y(T)$ and the target set. When the optimal point lies on the boundary of the target set, there exists a combination of the optimal policies. The combination of the optimal policies, the corresponding optimal trajectory, and the corresponding adjoint trajectory can be determined in the same way as in the statement 4. But in the case treated here the terminal condition of the differential system (3.5) (the transversality condition) is stated as follows:

   $[\eta(T)\ \eta_y(T)]$ is an inward normal vector of the target set at the boundary point $x_y^*(T)$.

Since we can prove these statements in the same way as in section 2, we omit the proofs.

3.3. Remarks on Advertising Policy

Advertising expenditures are similar in many respects to investment in plant and equipment. Advertising expenditures affect the demand function for the product, and hence on the operating profit of the firm. By adding new custmers and by altering the tastes and preferences of consumers such expenditures may shift the demand function and may change the shape of the function.

In order to represent the effects of current and past advertising expenditures we may define a "stock" in dollar term. We may call this stock "goodwill" according to Nerlove and Arrow [9]. Let $z(t)$ denote goodwill at time t. Then we may define the operating profit function by $\Pi[x(t), y(t), z(t), t]$, where $x(t)$ is capital stock and $y(t)$ is R & D fund.

Advertising expenditures in the past may be considered to contribute less, and so it may be said that goodwill "depreciates". Assuming depreciation at a fixed exponential rate δ_z the evolution of goodwill is determined by

$$\dot{z}=I_z(t)-\delta_z z(t)$$

$$z(0)=z_0,$$

where $I_z(t)$ is the rate of "gross investment in advertising" (current advertising outlay) and z_0 is the initial state of goodwill.

We assume that $I_z(t)$ is a Lesbegue measurable function on $0\leqq t\leqq T$, and that the values of $I_z(t)$ must satisfy the constraint

$$0\leqq I_z(t)\leqq I_{z\,\max} \qquad (0\leqq t\leqq T)$$

Let $\Omega_z\subset R^1$ denote the set of all I_z that satisfy the constraint.

Now the problem is formulated as follows.

Find the combination of

(1) the optimal investment policy $I(t)\subset\Omega$ on $0\leqq t\leqq T$ that steers the initial state of capital stocks x_0 to a terminal state in some prescribed target set in the x-space R^n by the process

$$\dot{x}=I(t)-\delta x(t),$$

(2) the optimal research and development policy $I_y(t)\subset\Omega_y$ on $0\leqq t\leqq T$ that steers the initial state of R & D fund y_0 to a terminal state in some prescribed target set in the y-space R^1 by the process

$$\dot{y}=I_y(t)-\delta_y y(t),$$

and

(3) the optimal advertising policy $I_z(t)\subset\Omega$ on $0\leqq t\leqq T$ that steers the initial state of goodwill z_0 to a terminal state in some prescribed target set in the z-space R^1 by the process

$$\dot{z}=I_z(t)-\delta_z z(t),$$

where the combination of the optimal policies is the combination of $I(t)$, $I_y(t)$ and $I_z(t)$ for which the objective functional

$$J(I)=\int_0^T \{S[I(t),\ t]+I_y(t)+I_z(t)-\Pi[x(t),\ y(t),\ z(t),\ t]\}e^{-rt}\ dt$$

achieves the least possible value.

Under some plausible assumptions on the functions Π and S we can solve this problem by the same method as used in the previous sections.

References

[1] Arrow, K. J., "Optimal Capital Adjustment," in *Studies in Applied Probability and Management Science*, K. J. Arrow, S. Karlin and H. Scarf (eds), Stanford: Stanford Univ. Press, 1962.

[2] Arrow, K. J., "Optimal Capital Policy with Irreversible Investment," in *Value, Capital, and Growth*, J. N. Wolfe (ed), Edinburgh: University Press, 1968.

[3] Arrow, K. J., M. J. Beckmann and S. Karlin, "The Optimal Expansion of the Capacity of a Firm," in *Studies in the Mathematical Theory of Inventory and Production*, K. J. Arrow, S. Karlin and H. Scarf (eds), Stanford: Stanford Univ. Press, 1958.

[4] Coddington, E. A., and N. L. Levinson, *Theory of Ordinary Differential Equations*, New York: McGraw-Hill Book Co., Inc., 1955.

[5] Dhrymes, P. J., "On Optimal Advertising, Capital and Research Expenditures under Dynamic Conditions," *Economica*, Vol. 29, N. S., Aug. 1962.

[6] Lee, E. B. and L. Markus, "Optimal Control for Nonlinear Processes," *Archive for Rational Mechanics and Analysis*, Vol. 8, pp. 36–58, 1961.

[7] Lee, E. B. and L. Markus, *Foundations of Optimal Control Theory*, New York: John Wiley & Sons, Inc., 1967.

[8] Maeda, H., "Optimal Investment Policies of the Firm: Their Existence and Optimality Conditions," (in Japanese), *The Journal of Economic Studies*, Vol. 13, Dec. 1968.

[9] Nerlove, H. and K. J. Arrow, "Optimal Advertising Policy under Dynamic Conditions," *Economica*, Vol. 29, N. S., May 1962.

[10] Schmaedeke, W. W., "Existence Theory of Optimal Controls," in *Advances in Control Systems: Theory and Applications, Vol. 3*, C. T. Leondes (ed), New York: Academic Press Inc., 1965.